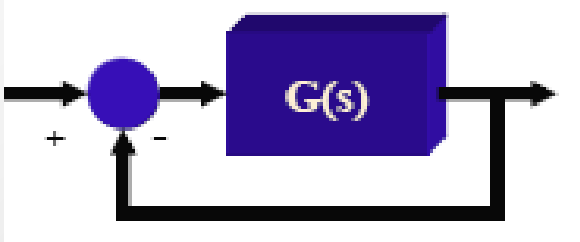
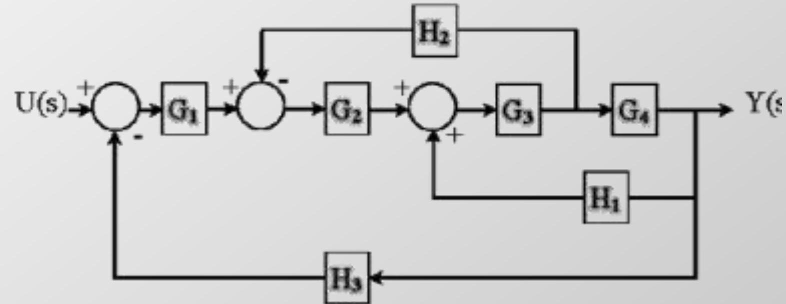
Mathematical Models of Systems



Outlines

- Block Diagram Models
- Basic Elements of Signal Flow Graph
- Basic Properties
- Definitions of Signal Flow Graph Terms
- Mason Theory
- Examples

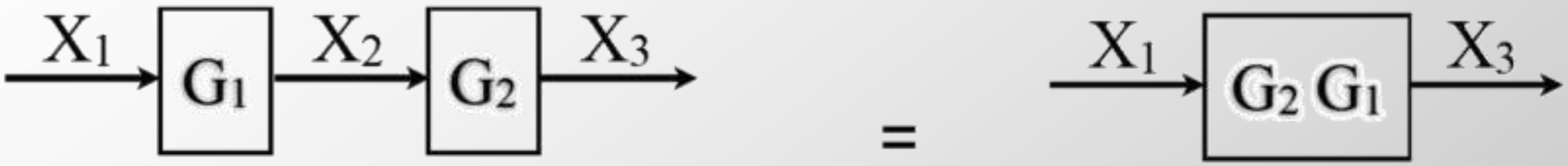
Block Diagram Models

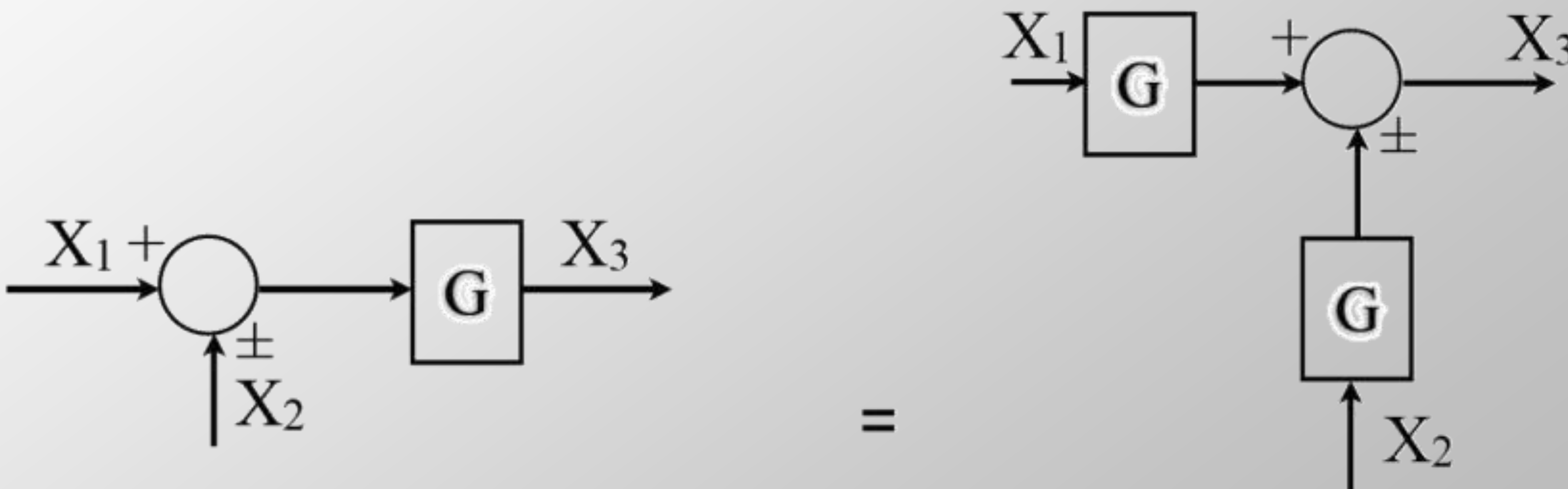


- Visualize input output relations
- Useful in design and realization of (linear) components
- Helps understand flow of information between internal variables.
- Are equivalent to a set of linear algebraic equations

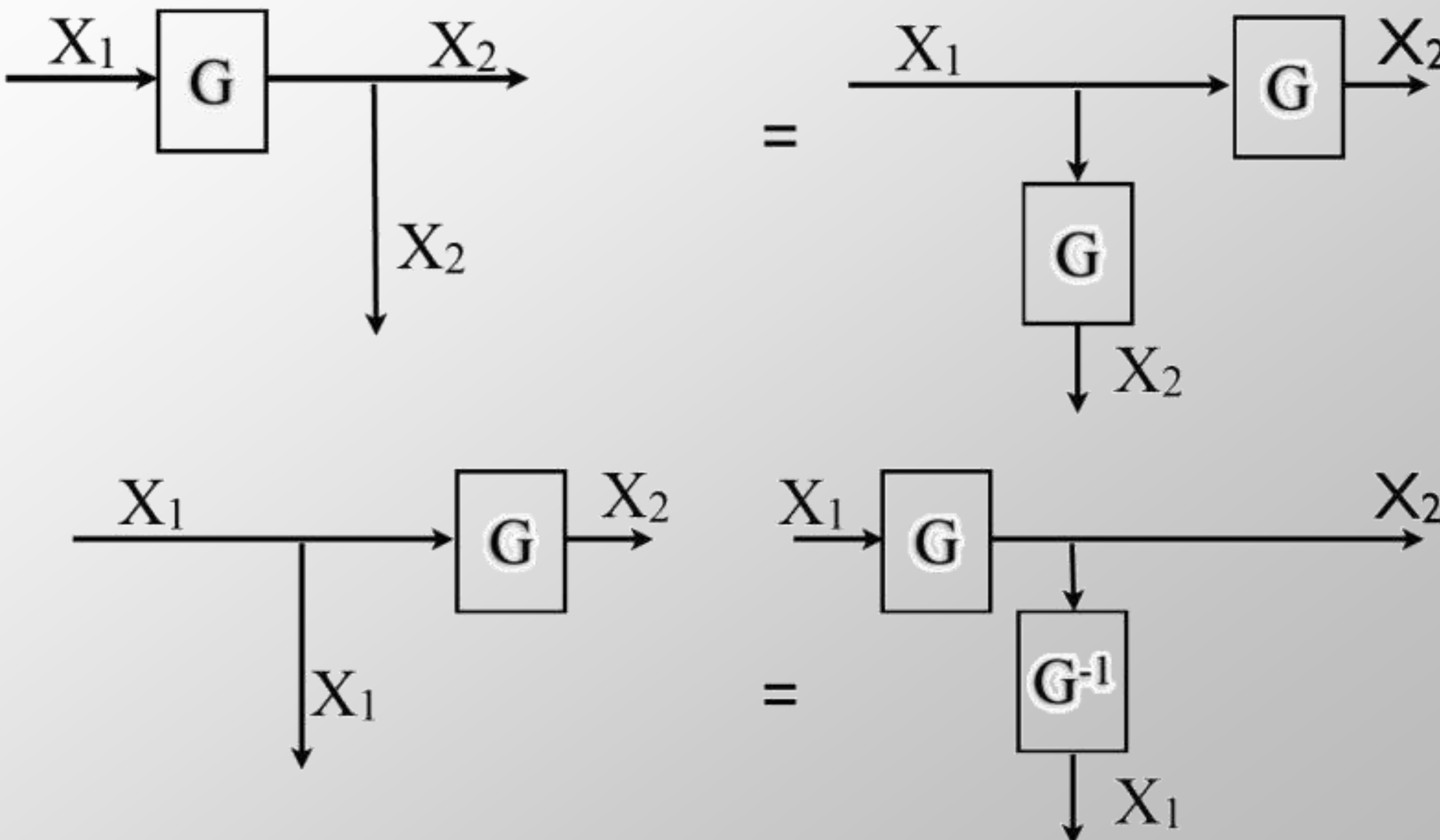
  (of rational functions).
- Mainly relevant where there is a cascade of information flow

Block Diagram Manipulation Rules





Block Diagram Manipulation Rules



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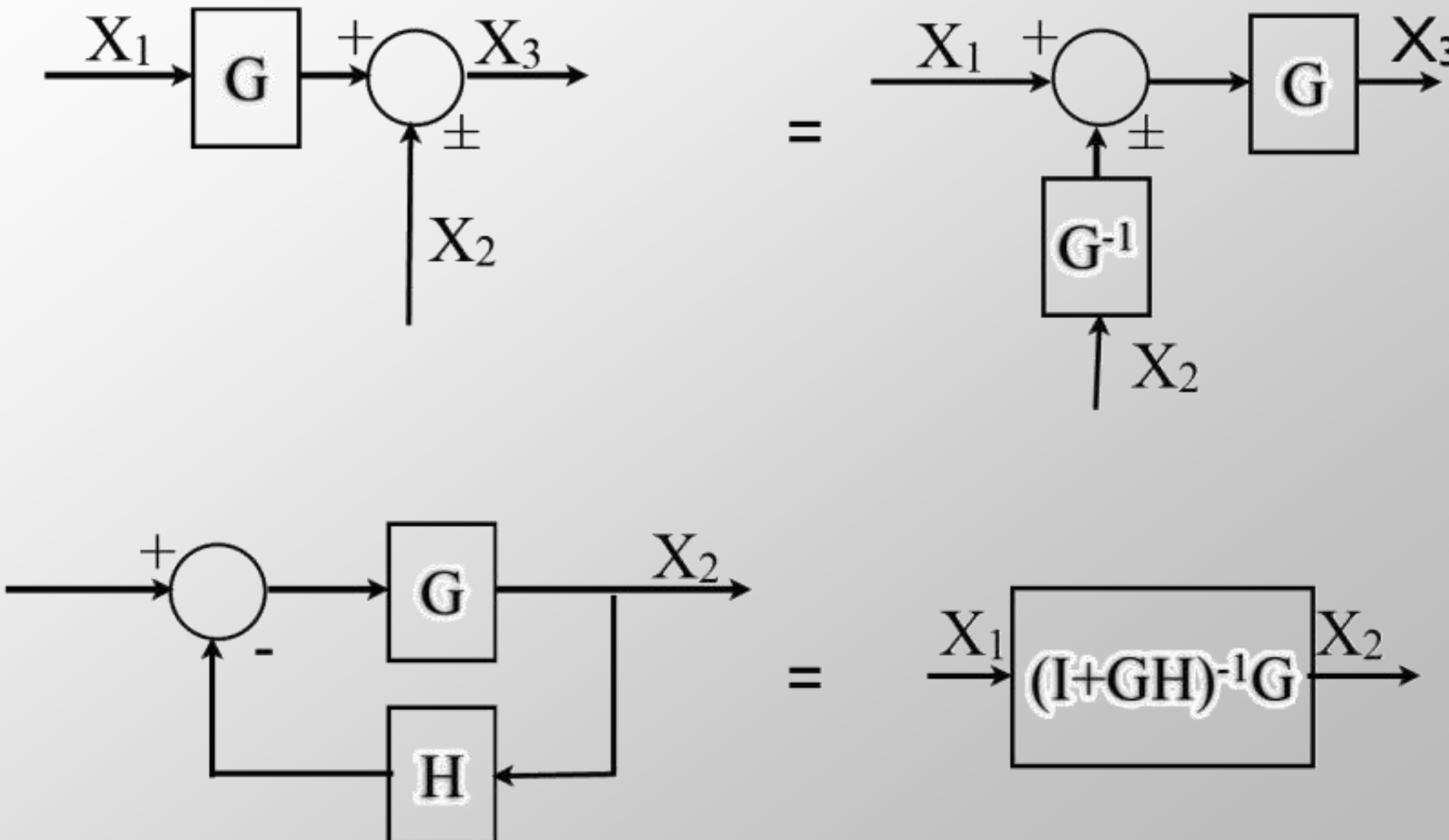
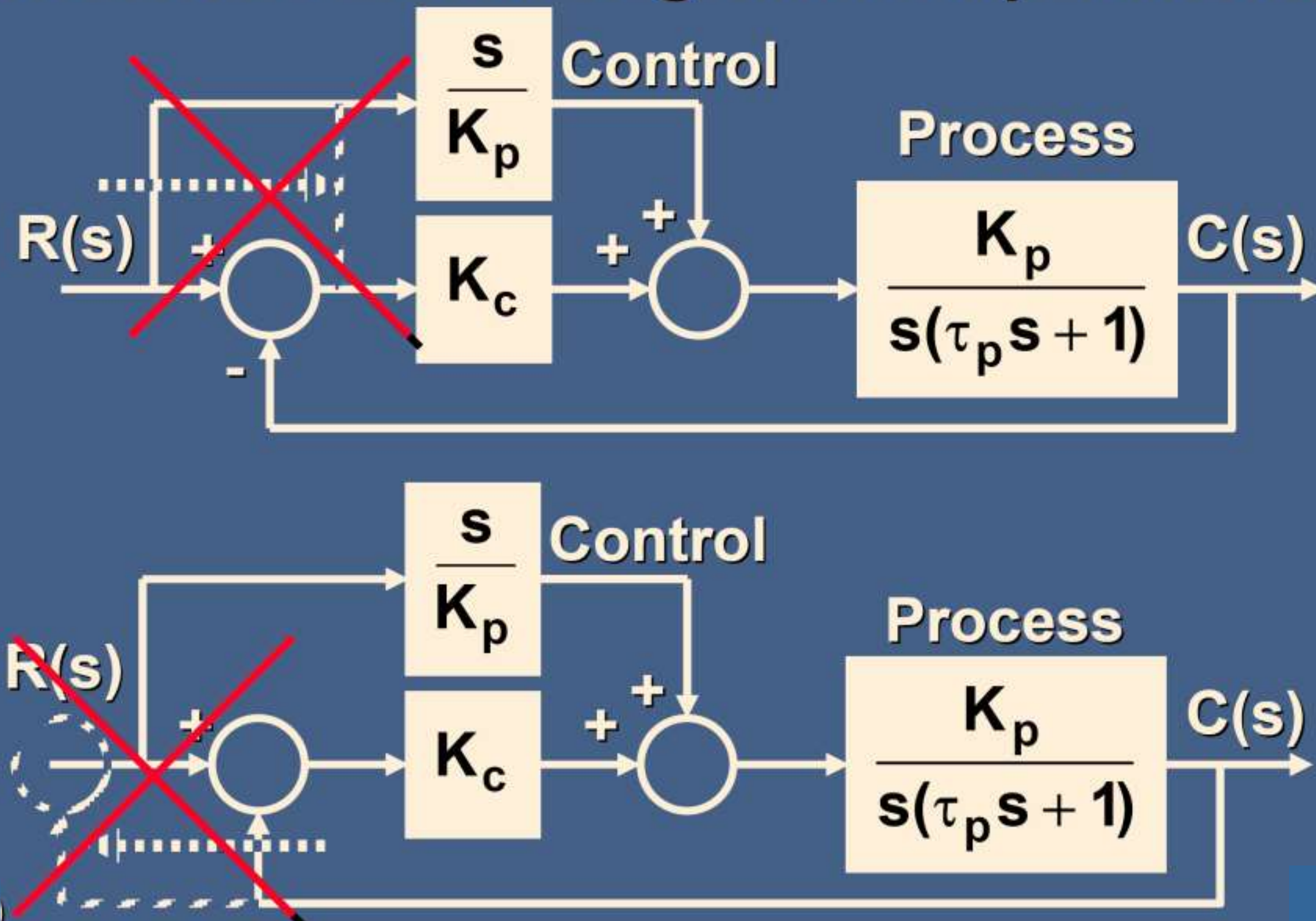
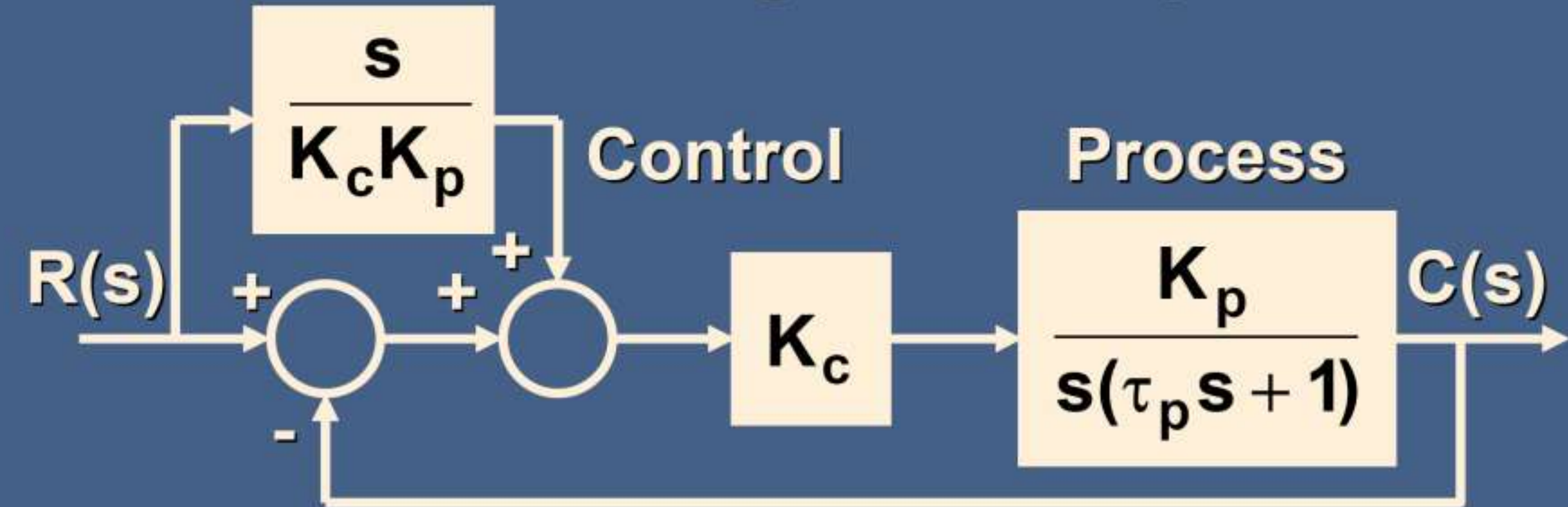
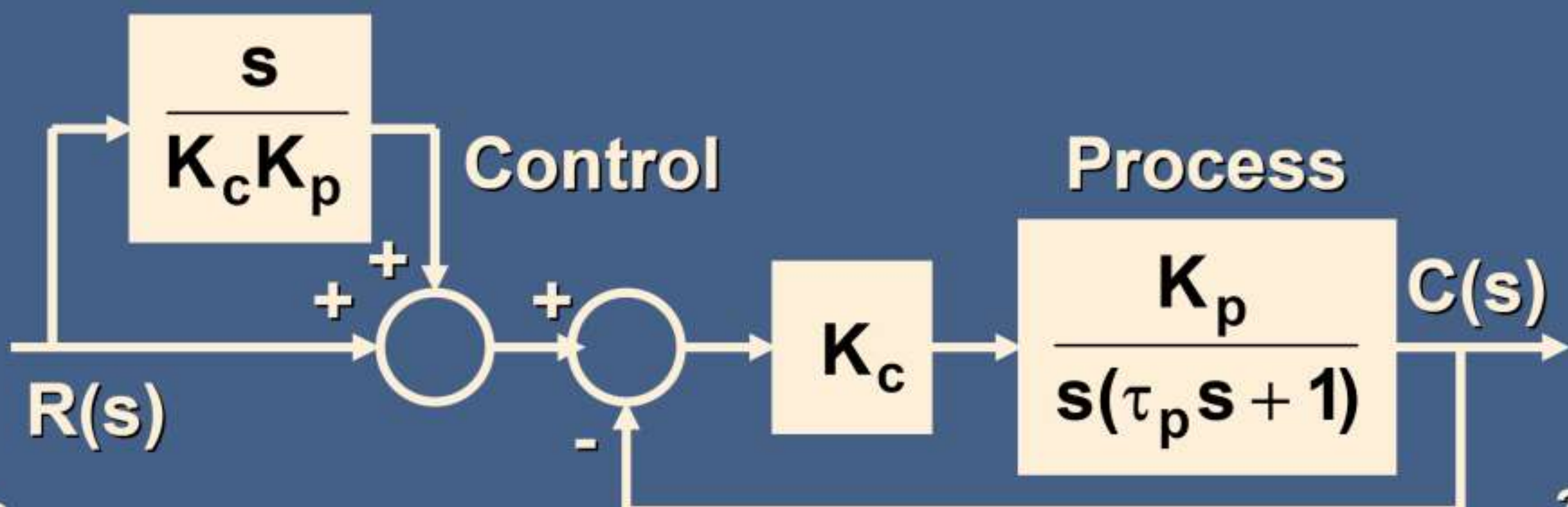


Incorrect Block Diagram Manipulations

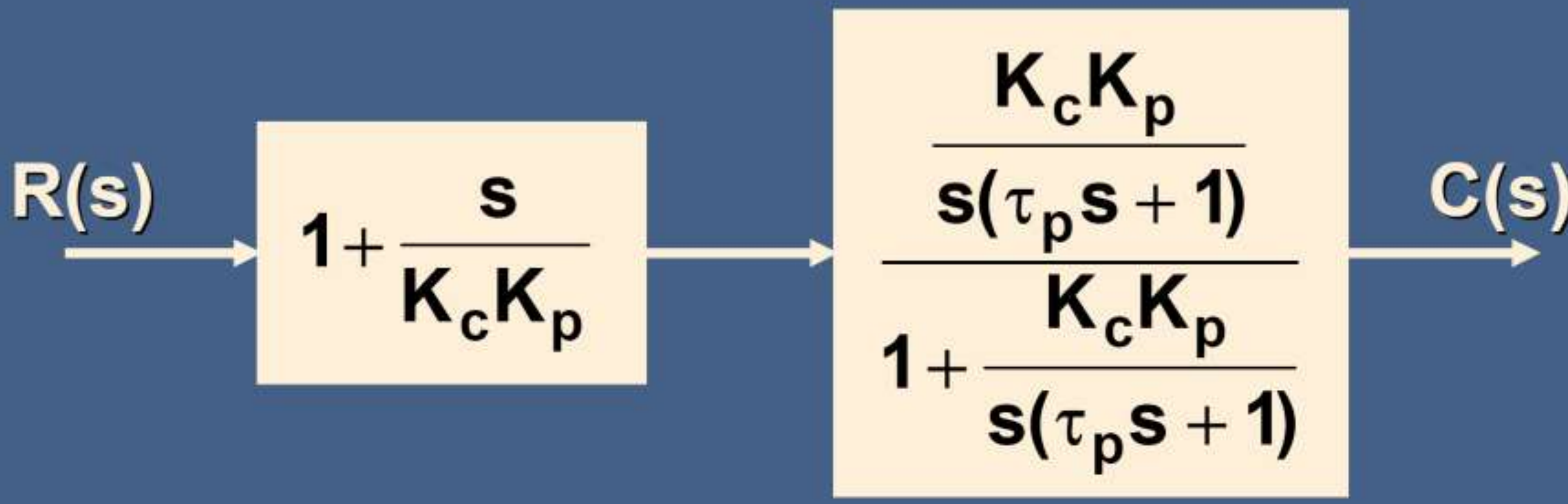


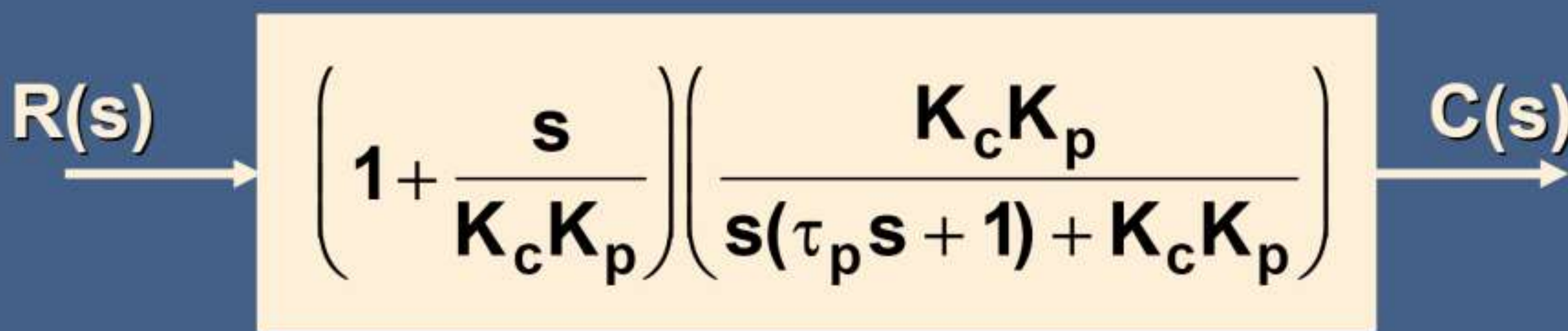
Correct Block Diagram Manipulations





Block Diagram Reduction





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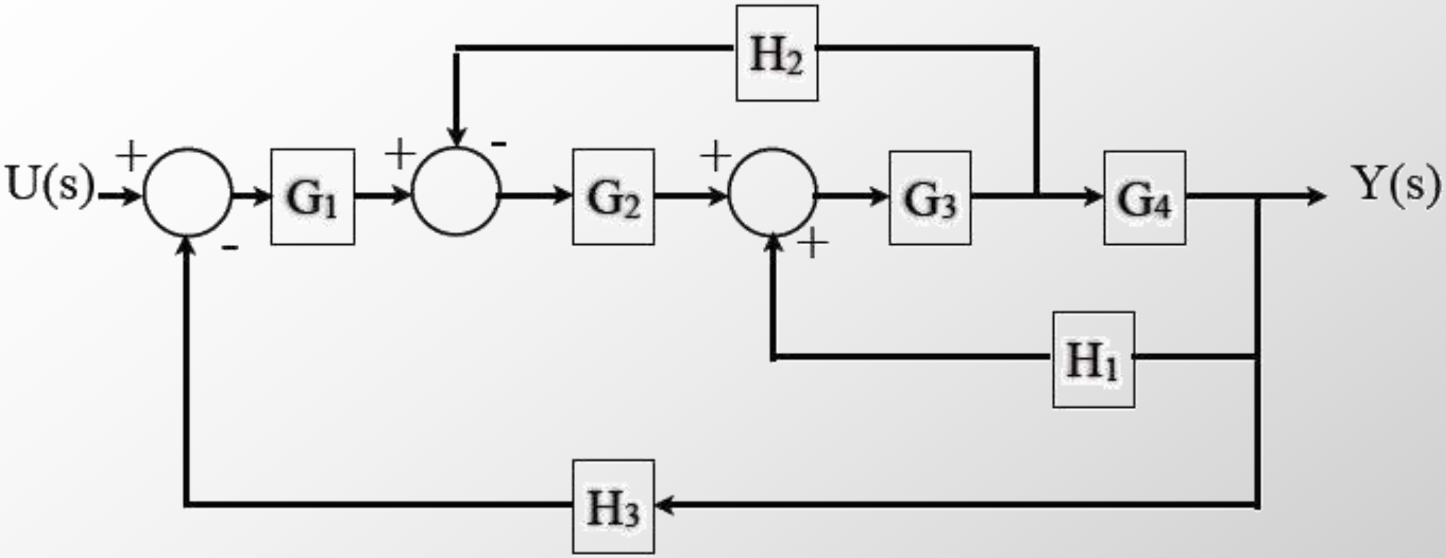

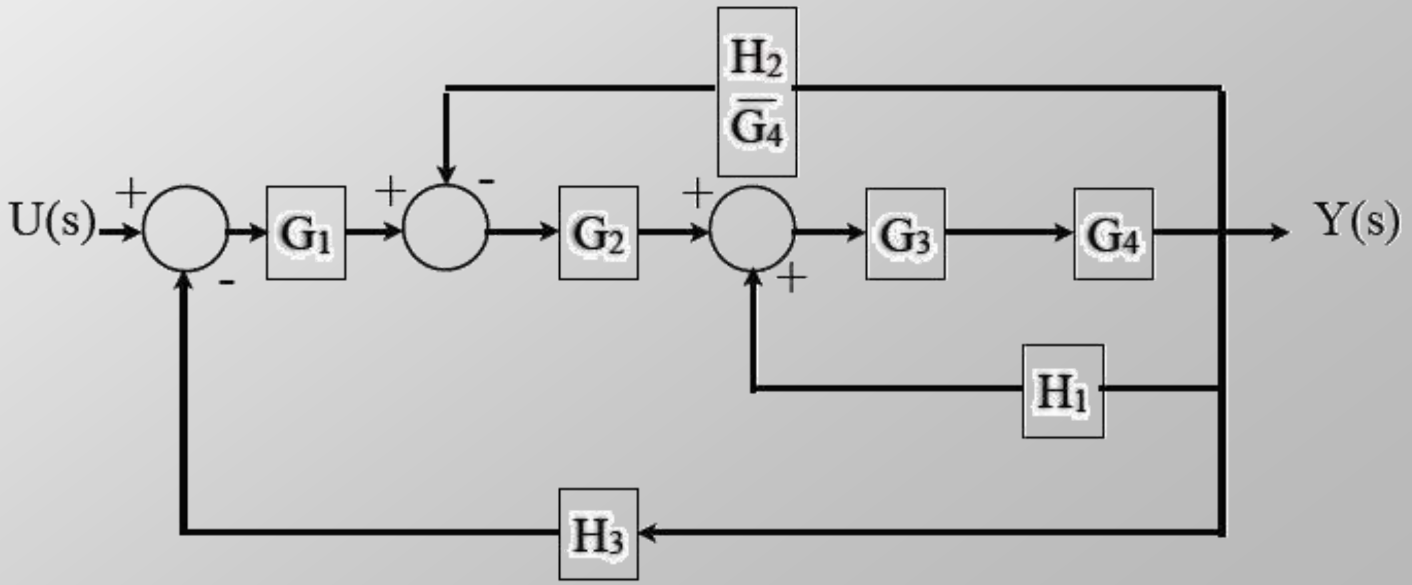

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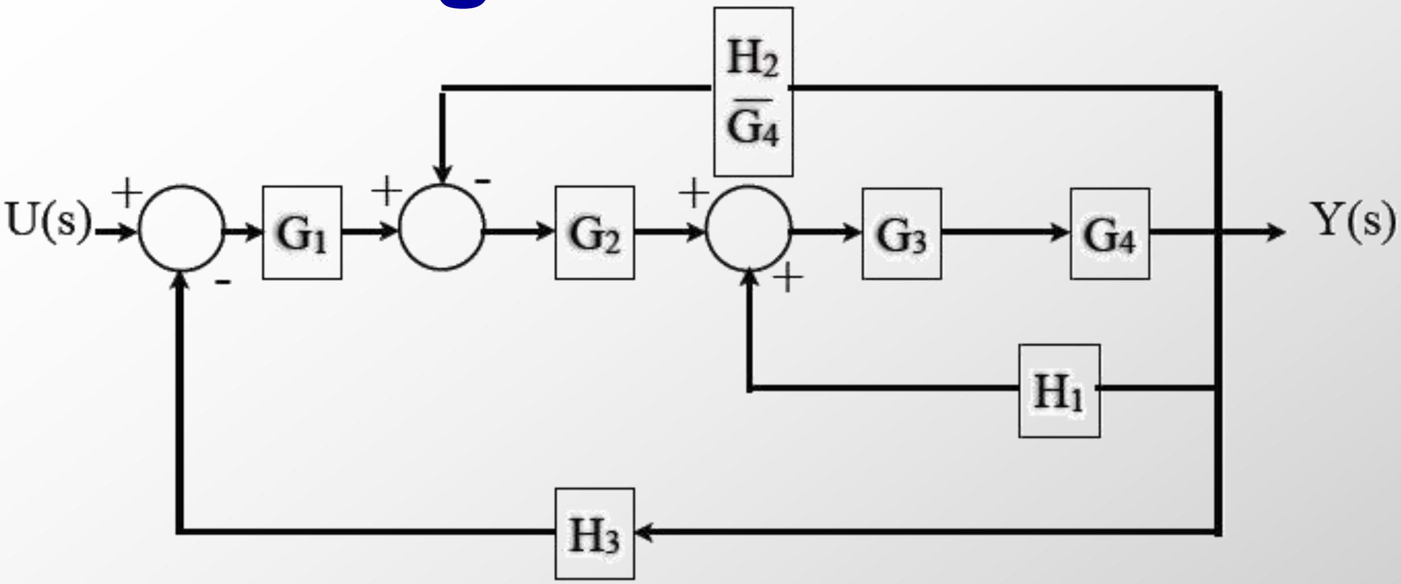

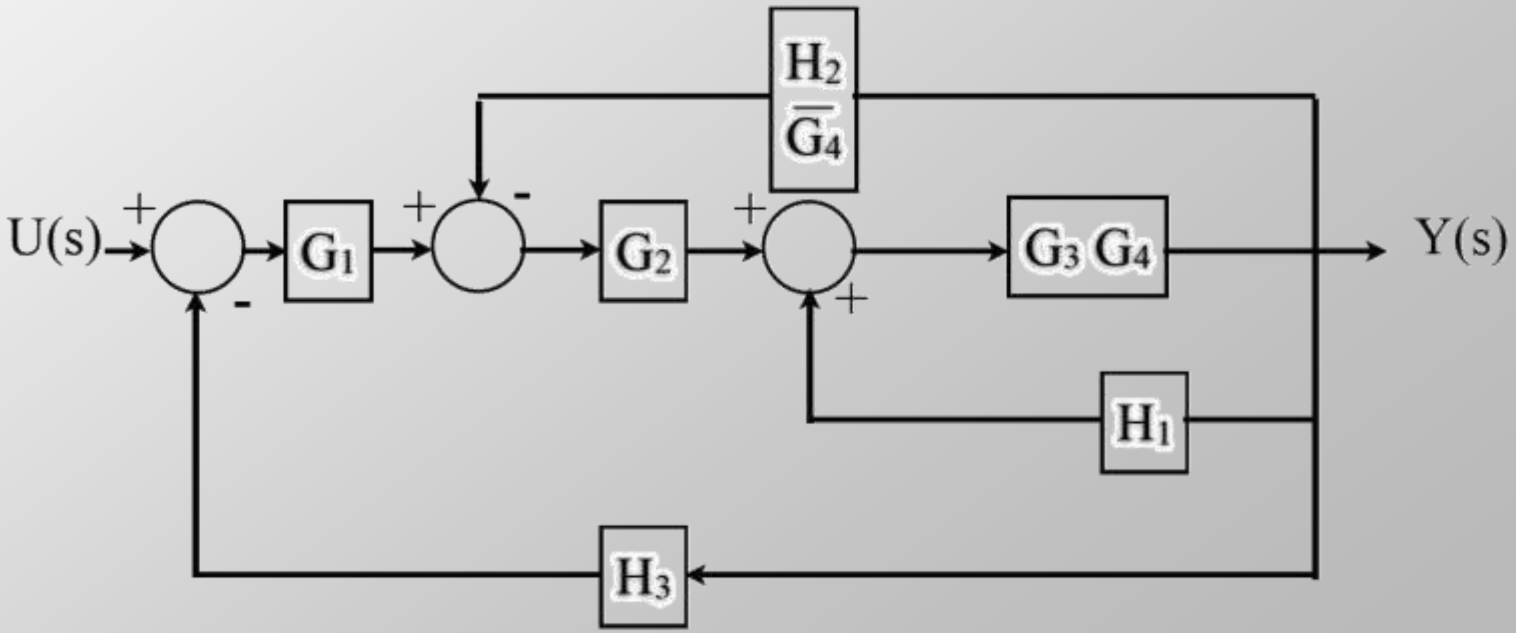

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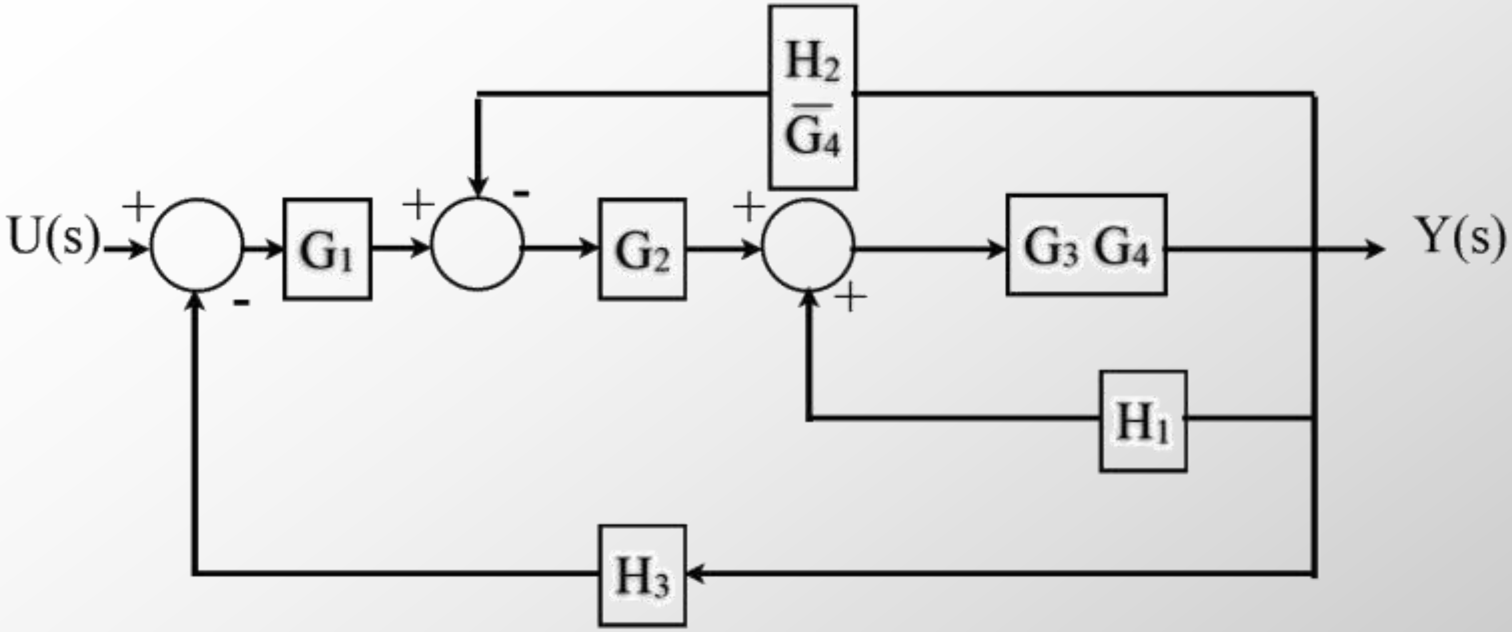

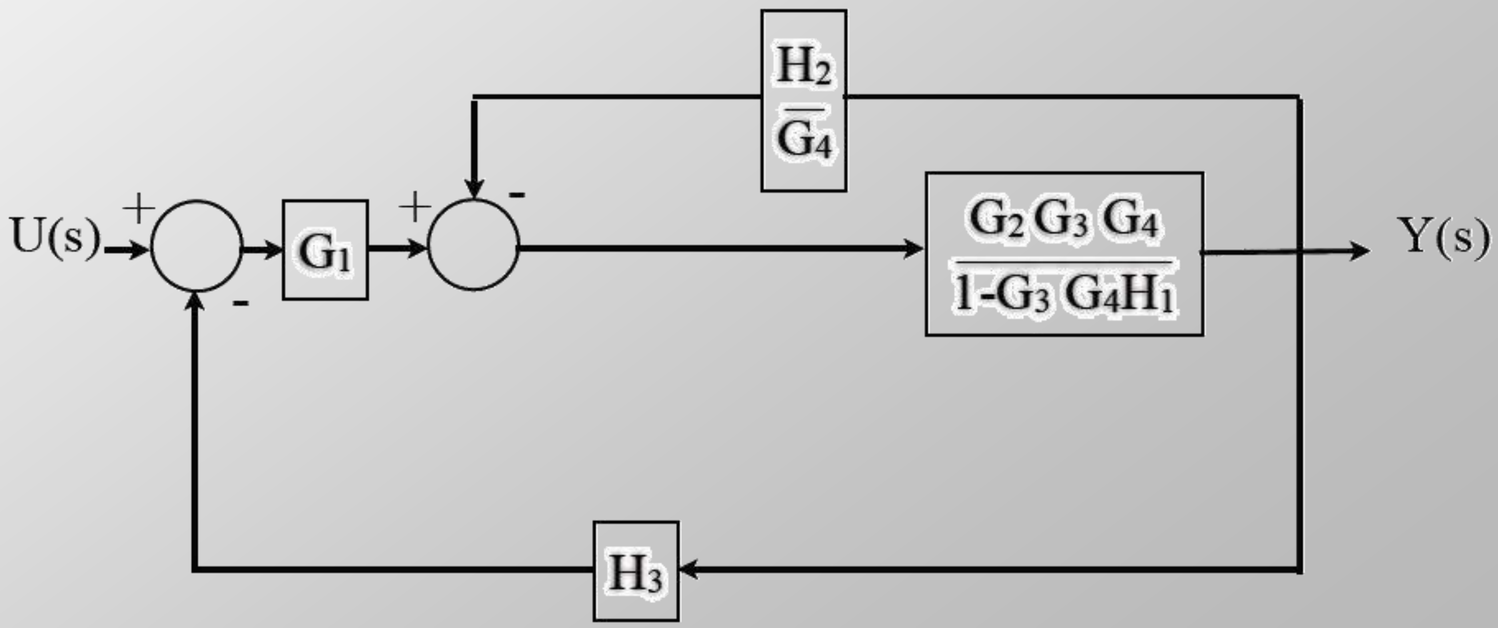

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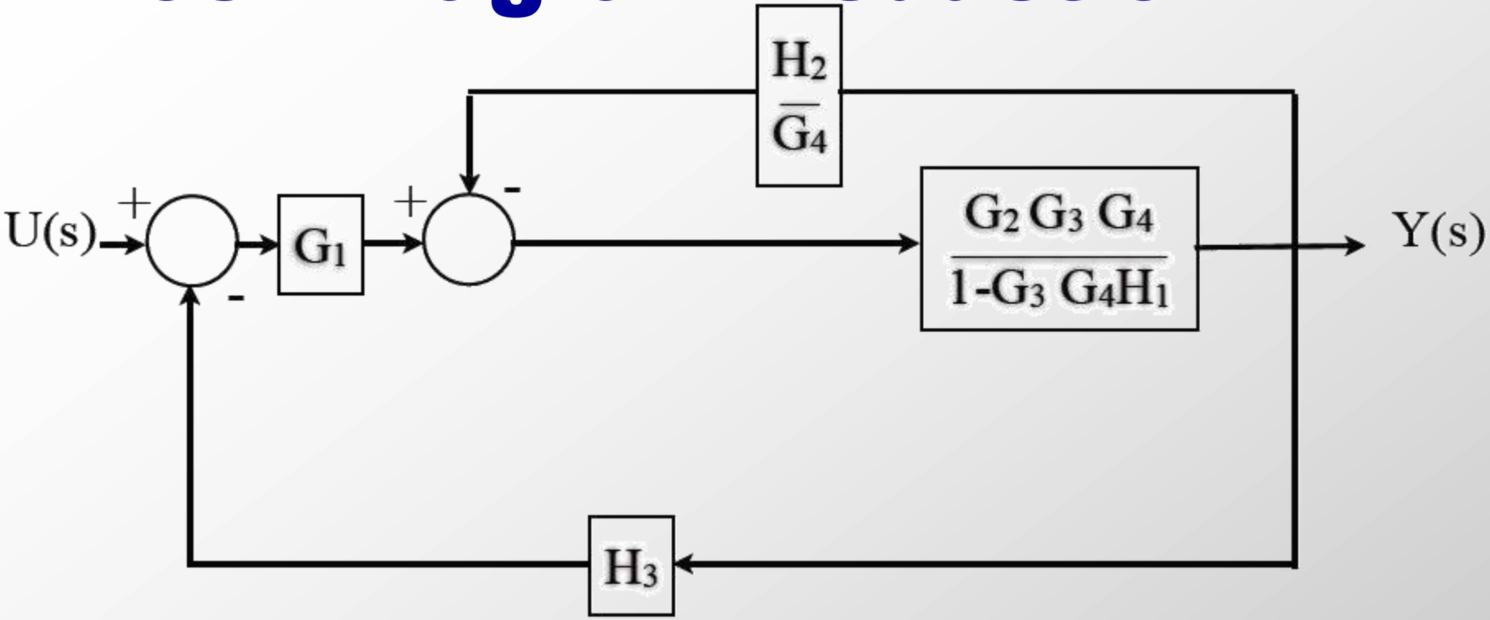

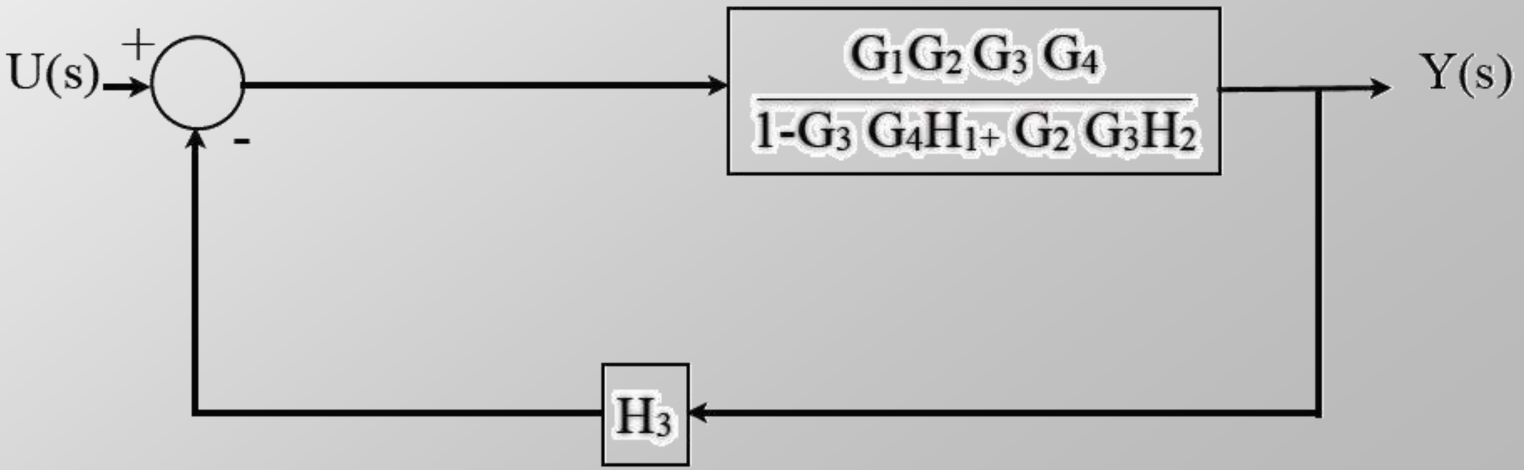

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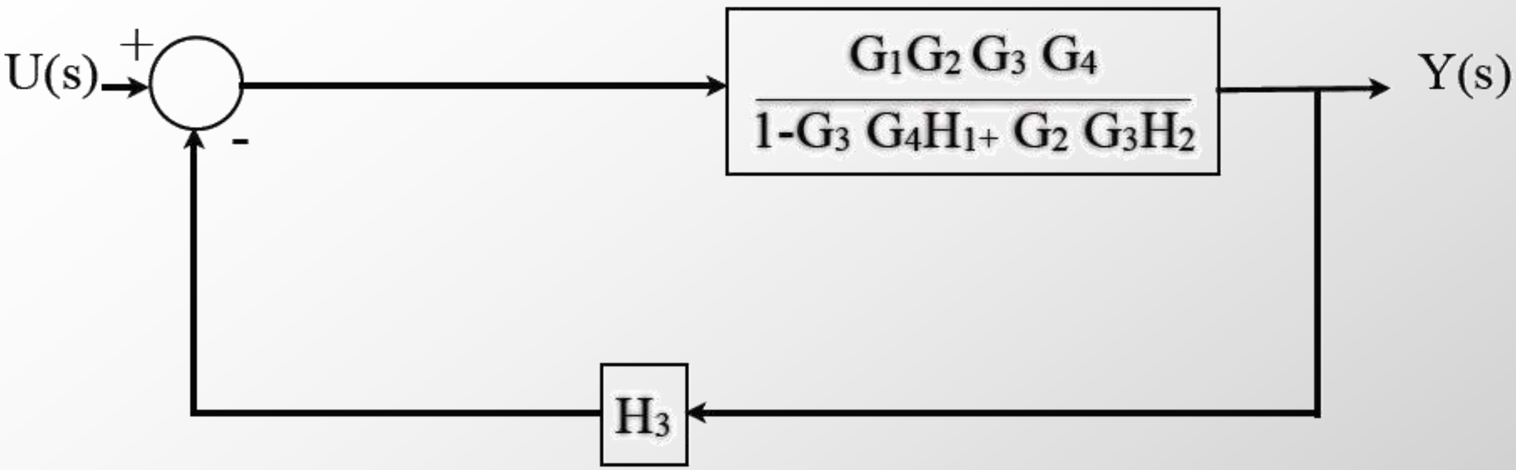
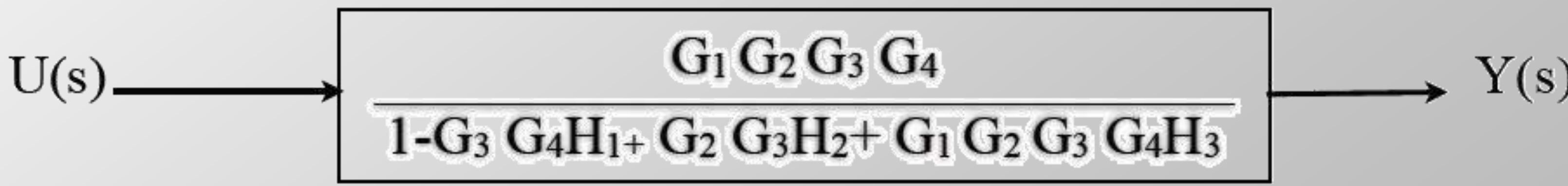




Signal Flow Graphs

- Alternative to block diagrams
- Do not require iterative reduction to find transfer functions (using Mason's gain rule)
- Can be used to find the transfer function between any two variables (not just the input and output).
- Look familiar to computer scientists (?)

Signal-flow graph

From Wikipedia, the free encyclopedia

A signal-flow graph (SFG) is a special type of block diagram, constrained by rigid mathematical rules, that is a graphical means of showing the relations among the variables of a set of linear algebraic relations. Nodes represent variables, and are joined by branches that have assigned directions (indicated by arrows) and gains. A signal can transmit only in the direction of the arrow.

Utility of Signal Flow Graphs

- Alternative to block diagram approach

  - may be better for complex systems

  - good for highly interwoven systems

  - system variables represented as nodes

  - branches (lines) between nodes show relationships between system variables

- The "flow graph gain formula" (Mason) allows the system transfer function to be directly computed without manipulation or reduction of the diagram.

Example 1 Simple amplifier

$$y_1 = a_{11}x_1$$

$$x_1 = y_1 / a_{11}$$

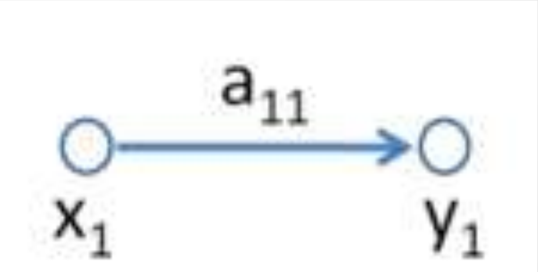


Figure 1: Simple SFG

Amplification of a signal x_1 to become a larger output y_2 by an amplifier with gain a_{11} is described mathematically by:

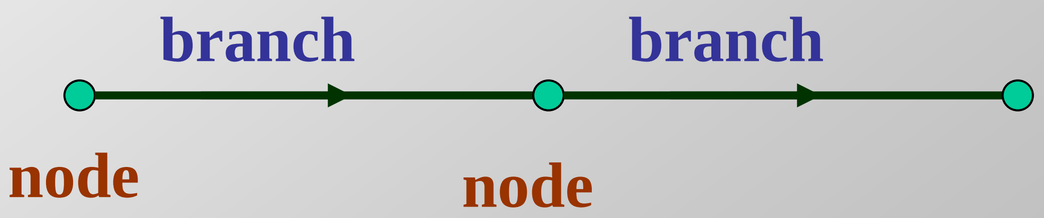
which becomes the signal flow graph of Figure 1.

Although this equation is represented by the SFG of Figure 1, the algebraically equivalent relation

is *not* considered to be implied by Figure 1. That is, the SFG is unilateral, sometimes emphasized by calling x_1 a "cause" and y_1 an "effect", or by calling x_1 an "input" and y_1 an "output".

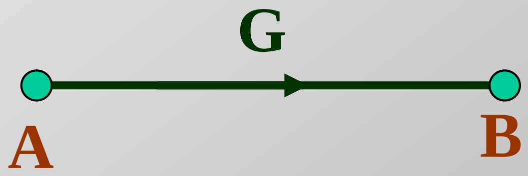
Basic Elements of Signal Flow Graph

- **A** **Signal flow graph** is a diagram consisting of **nodes** that are connected by several directed **branches**.



Basic Elements of Signal Flow Graph

- A **node** is used to represent a variable (inputs, outputs, other signals)
- A **branch** shows the dependence of one variable (**node**) on another variable (**node**)
  – Each **branch** has GAIN and DIRECTION
  – A signal can transmit through a **branch** only in the direction of the arrow
  – If gain is not specified gain =1



B = G A

Example 2 Two-port network

$$y_1 = a_{11}\, x_1 + a_{12}\, x_2$$

$$y_2 = a_{21}\, x_1 + a_{22}\, x_2\ ,$$

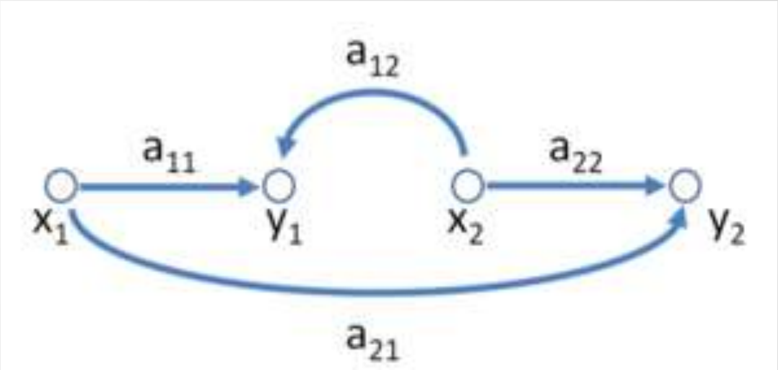


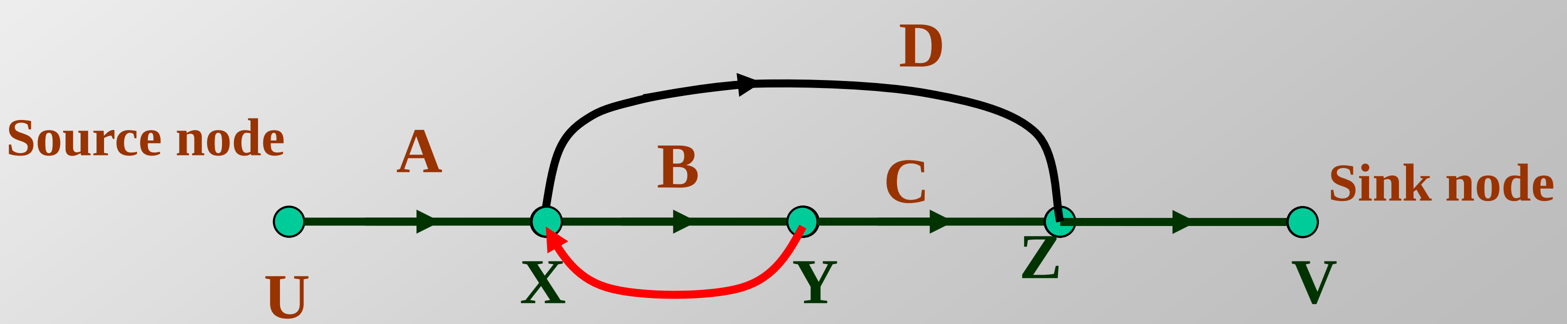
Figure 2: Two-port SFG

The two coupled equations below can represent the current-voltage relations in a two-port network:

which equations become the signal flow graph of Figure 2.

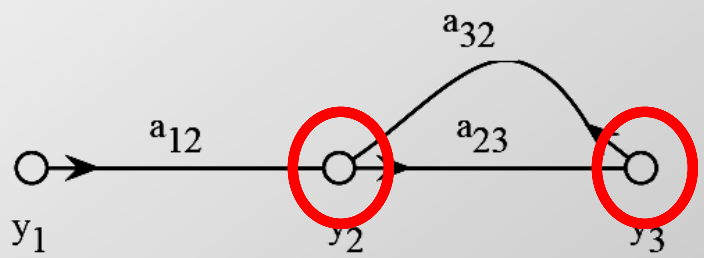
Nodes

- A **node** is used to represent a variable
- Source **node** (input node)
  - All braches connected to the node are leaving the node
  - Input signal is not affected by other signals
- Sink **node** (output node)
  - All braches connected to the node are entering the node
  - output signal is not affecting other signals

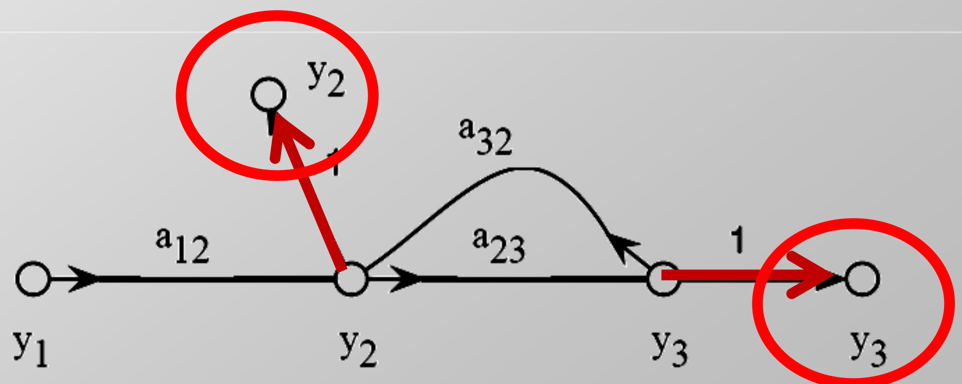


Input / Output

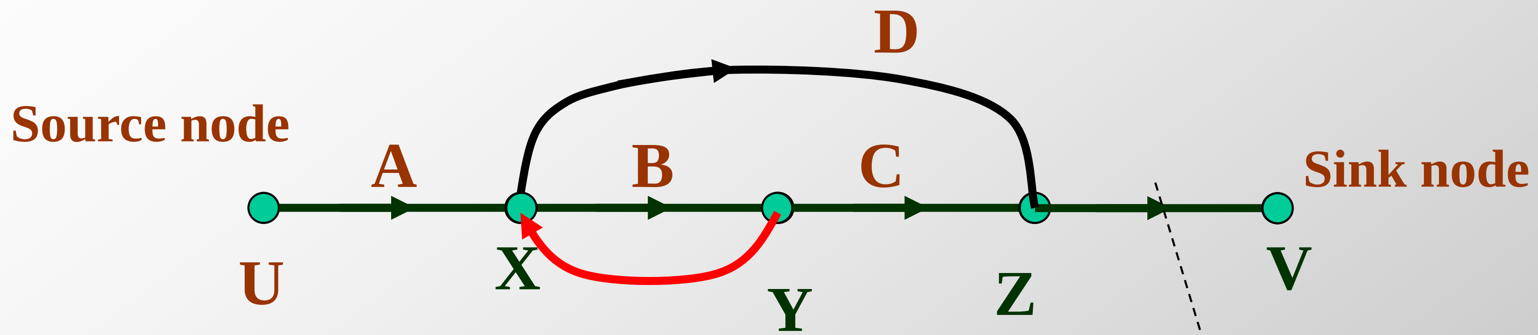
- Input (*source*) has only outgoing edges
- Output (*sink*) has only incoming edges



- *any* variable can be made into an output by adding a sink with "1" edge



Relationship Between Variables



Gain is not shown means gain=1

U (input)

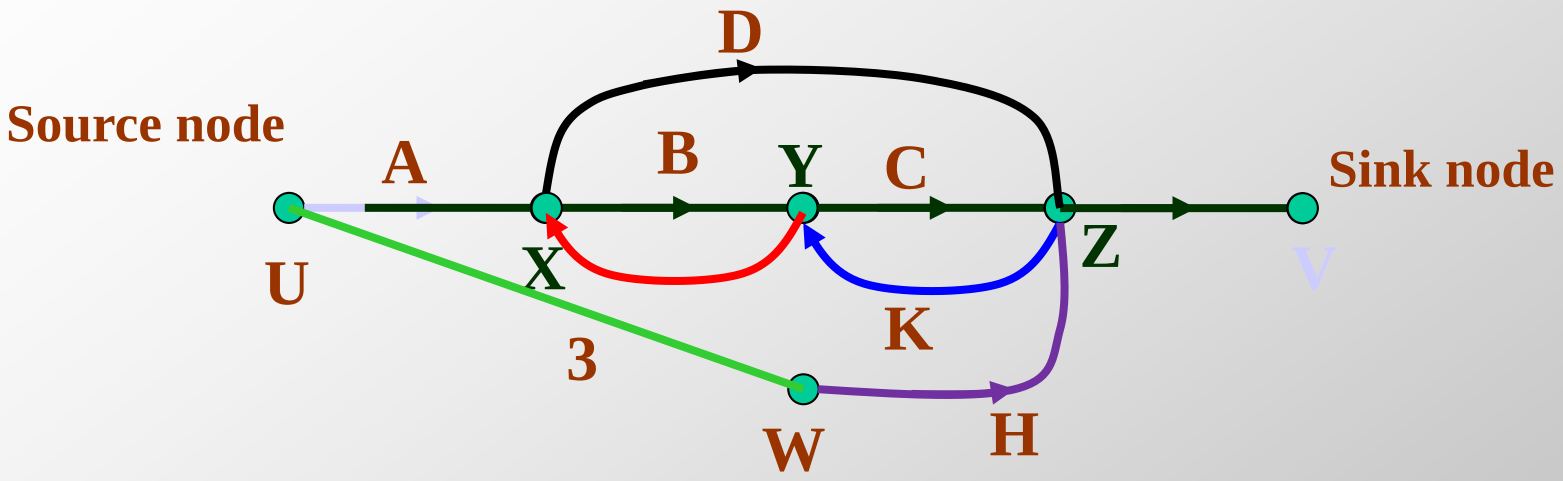
X=AU+Y

Y=BX

Z=CY+DX

V=Z (output)

Another Example



X=AU+Y

Y=BX+KZ

Z=CY+DX+HW

W=3U

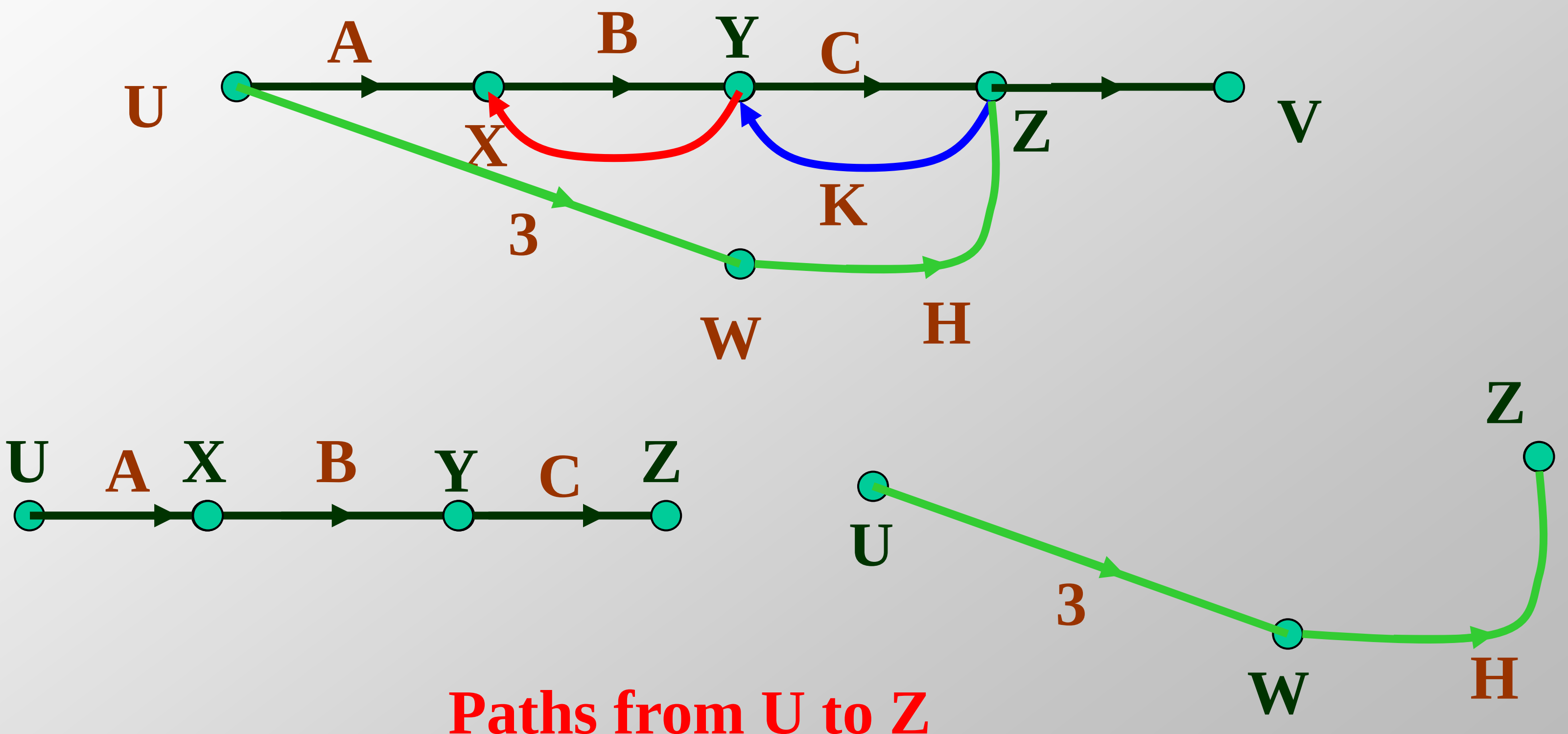
V=Z

Basic Properties

- Signal flow graphs applies to linear systems only
- Nodes are used to represent variables
- A branch from node X to node Y means that Y depends on X
- Value of the variable (node) is the sum of gain of branch * value of node
- Non-input node cannot be converted to an input node
- We can create an output node by connecting unit branch to any node

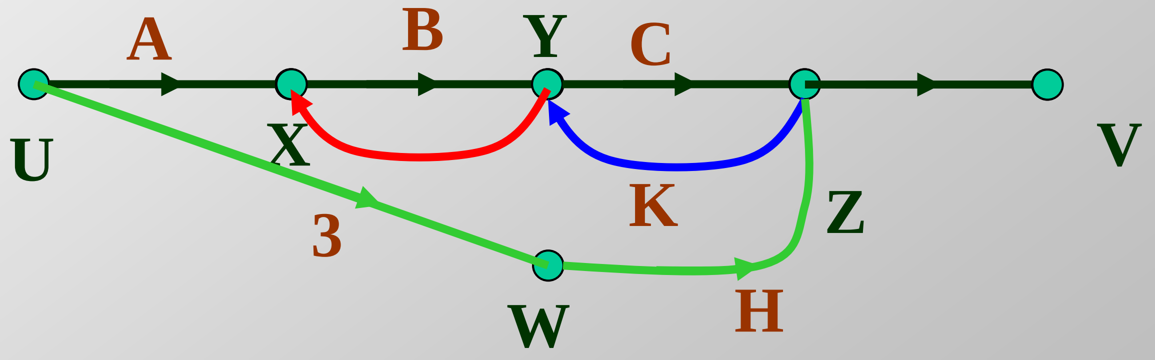
Terminology: Paths

- **A path:** is a branch or a continuous sequence of branches that can be traversed from one node to another node



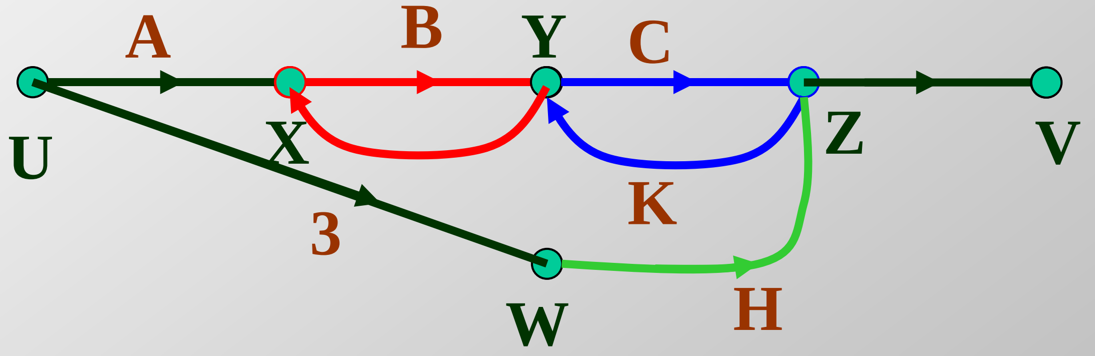
Terminology: Paths

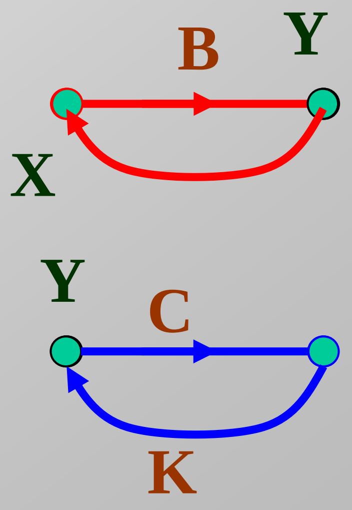
- **A path:** is a branch or a continuous sequence of branches that can be traversed from one node to another node
- **Forward path**: path from a source to a sink
- **Path gain**: product of gains of the braches that make the path



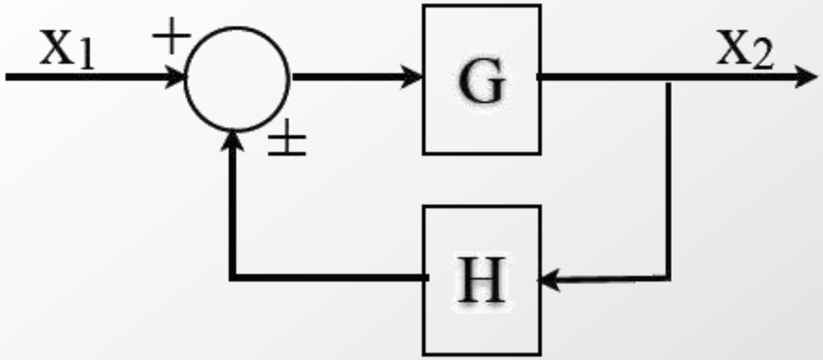
Terminology: loop

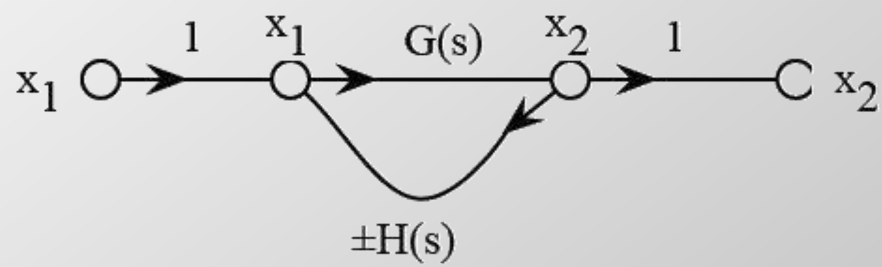
- A **loop:** is a closed path that originates and terminates on the same node, and along the path no node is met twice.
- **Nontouching loops**: two loops are said to be nontouching if they do not have a common node.





Block Diagram vs Signal Flow Graph





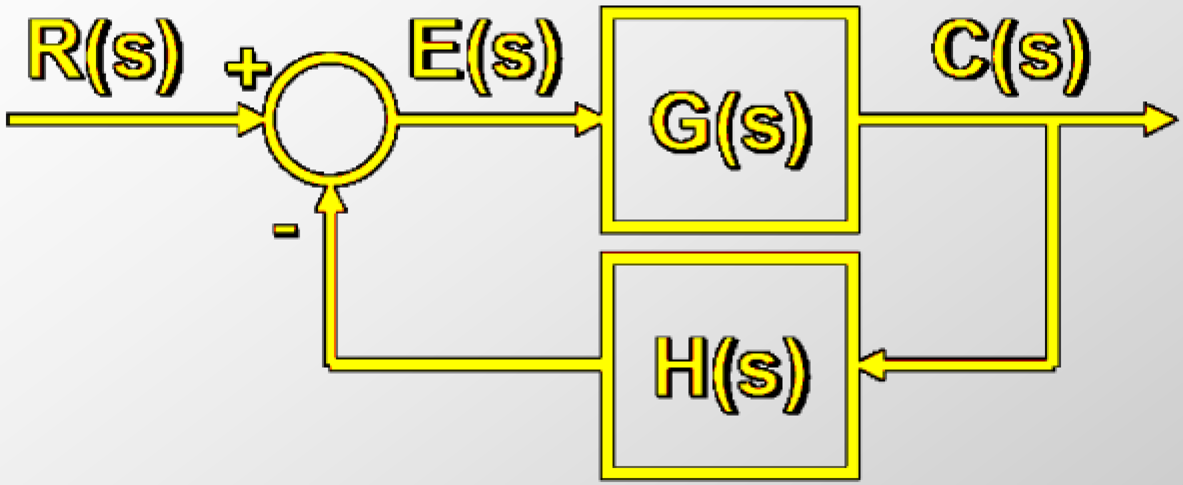
- Blocks ⇒ Edges (branches)

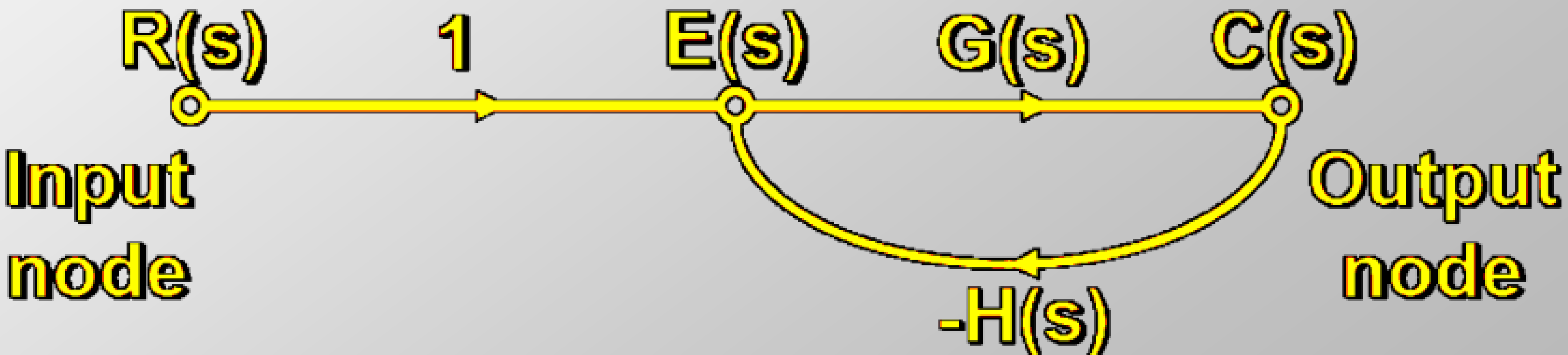
  (representing transfer functions)

- Edges + junctions ⇒ Vertices (nodes)

  (representing variables)

Basic Signal Flow Graph



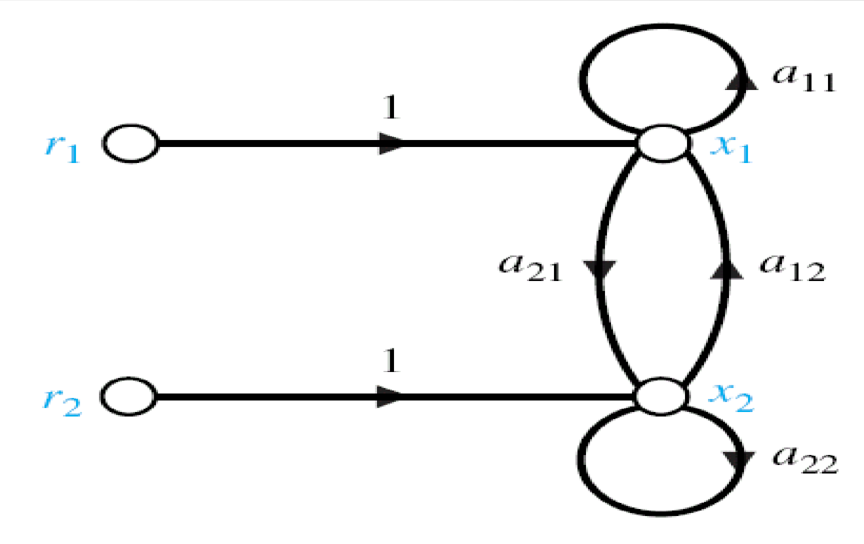


Algebraic Eq representation

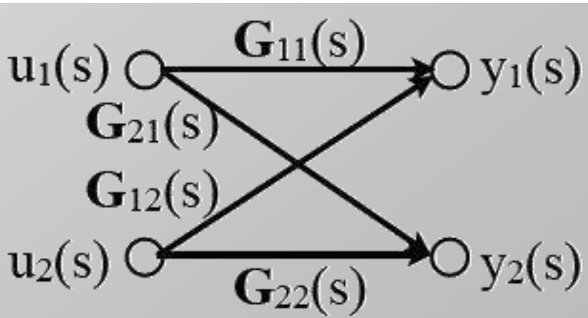
- $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{r}$

  $x_1 = a_{11}x_1 + a_{12}x_2 + r_1$

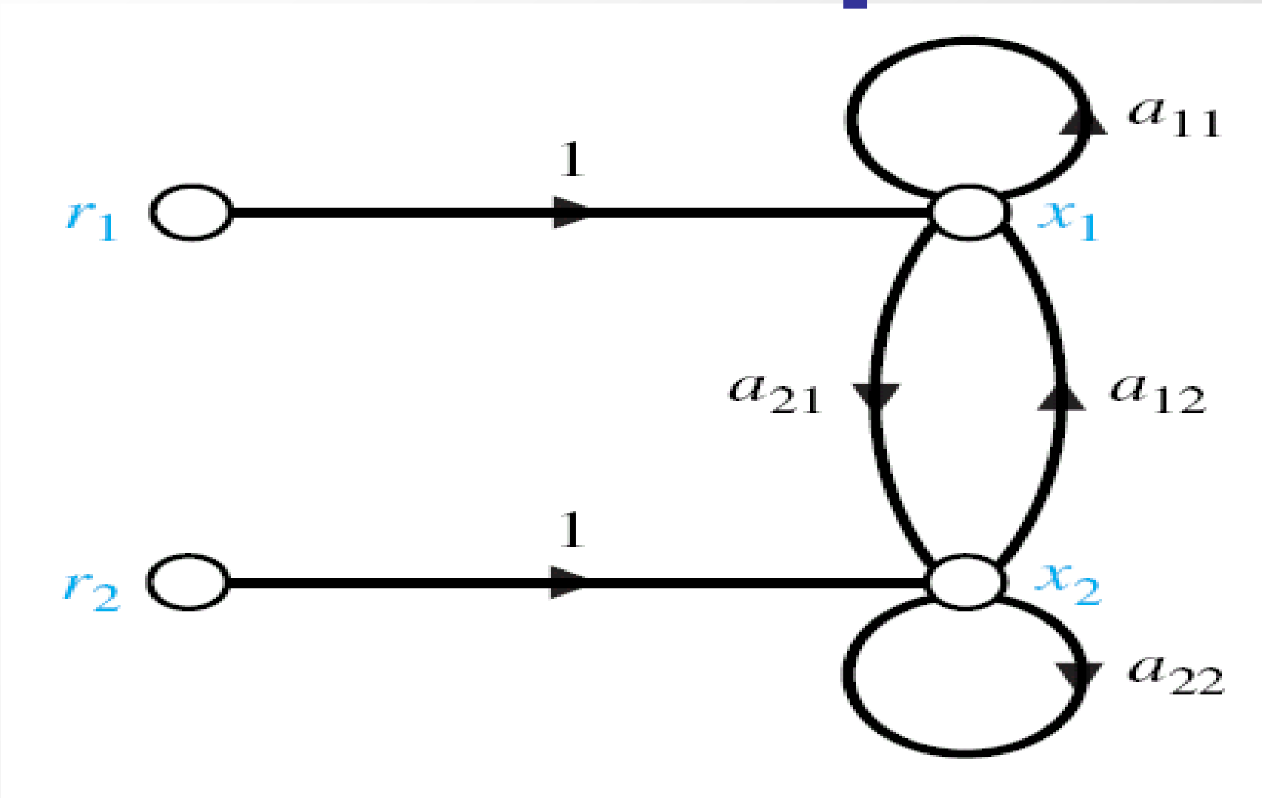
  $x_2 = a_{21}x_1 + a_{22}x_2 + r_2$



- $\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s)$



An example



$$a_{11} x_1 + a_{12} x_2 + r_1 = x_1$$

$$a_{21} x_1 + a_{22} x_2 + r_2 = x_2$$

An example...

$$(1 - a_{11})x_1 + (-a_{12})\ x_2 = r_1$$

$$(-a_{21})\ x_1 + (1 - a_{22})x_2 = r_2$$

- This have the solution

$$x_1 = (1 - a_{22})/\Delta\, r_1 + a_{12}/\Delta\, r_2$$

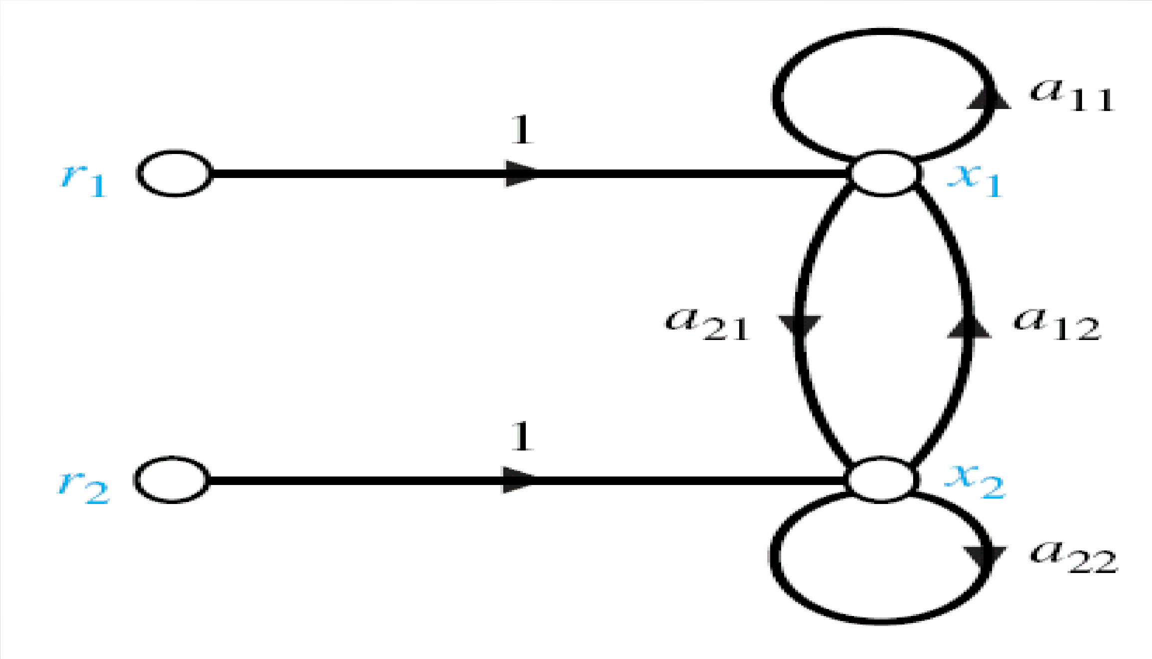
$$x_2 = (1 - a_{11})/\Delta\, r_2 + a_{21}/\Delta\, r_1$$

$$\Delta = 1 - a_{11} - a_{22} + a_{22}\, a_{11} - a_{12}\, a_{21}$$

An example

$$\Delta = 1 - a_{11} - a_{22} + a_{22}a_{11} - a_{12}a_{21}$$

- Self loops a_{11} , a_{22} , $a_{12}a_{21}$
- Product of non touching loops $a_{22}a_{11}$



SGF : in general

- The linear dependence (T_{ij}) between the independent variable x_i (input) and the dependent variable (output) x_j is given by Mason's SF gain formula
- $$T_{ij} = \frac{\sum_k P_{ijk} \Delta_{ijk}}{\Delta}$$
  $$P_{ijk} = k^{\text{th}} \text{ path from } \mathrm{x_i} \text{ to } \mathrm{x_j}$$
  $$\Delta = \text{ determinant of the graph}$$
  $$\Delta_{ijk} = \text{cofactor of the path } P_{ijk}$$

The determinant Δ

$$\Delta = 1 - \sum_{n=1}^{N} L_n + \sum_{m=1,q=1}^{M,Q} L_m L_q - \sum L_r L_s L_t \ldots$$

$$L_q \text{ is the loop gain}$$

- Or

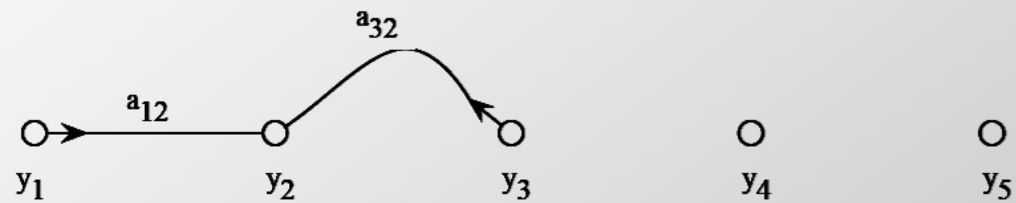
Δ=1 –(sum of all different loop gains) +(sum of the gain products of all combinations of 2 non-touching loops)

-(sum of the gain products of all combinations of 3 non-touching loops)…

- The cofactor Δ_{ijk} is the determinant with loops touching the kth path removed

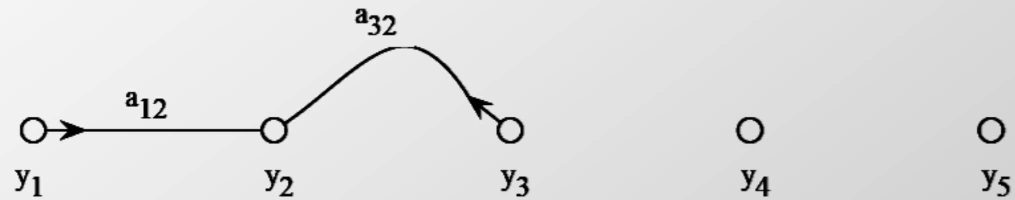
Another SFG Example

$y_2 = a_{12}y_1 + a_{32}y_3$

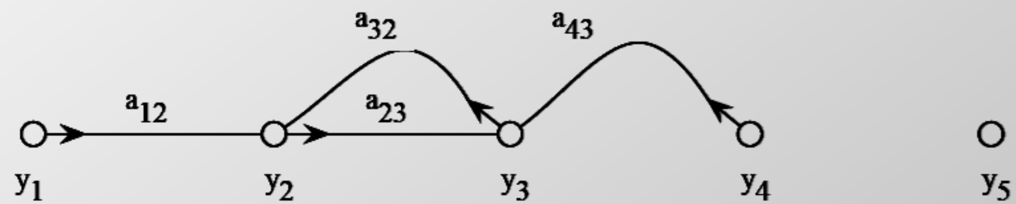


Another SFG Example

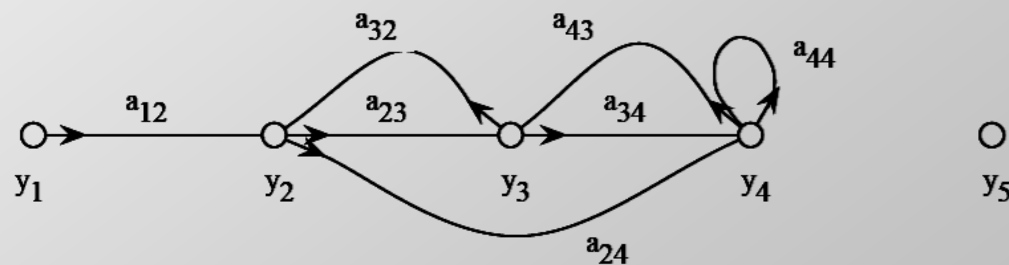
$y_2 = a_{12}y_1 + a_{32}y_3$



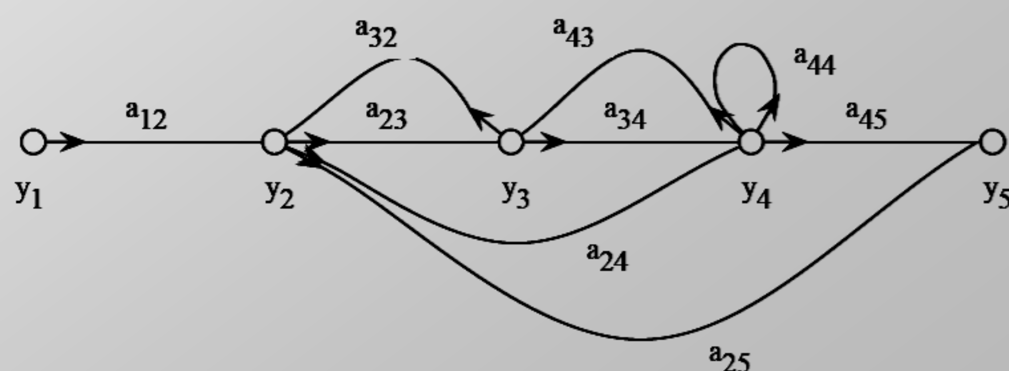
$y_3 = a_{23}y_2 + a_{43}y_4$



$y_4 = a_{24}y_2 + a_{34}y_3 + a_{44}y_4$



$y_5 = a_{25}y_2 + a_{45}y_4$

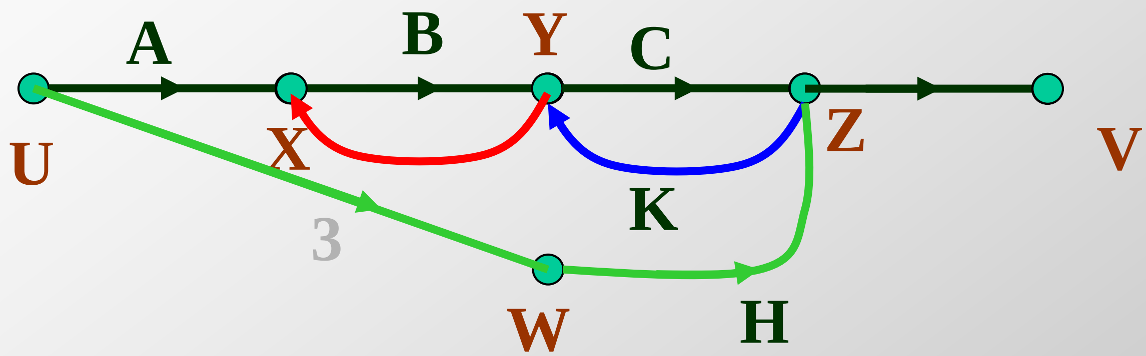


Definitions

- **Input:** (*source*) has only outgoing branches
- **Output:** (*sink*) has only incoming branches
- **Path:** (from node i to node j) has no loops.
- **Forward-path:** path connecting a source to a sink
- **Loop:** A simple graph cycle.
- **Path Gain:** Product of gains on path edges
- **Loop Gain:** Product of gains on loop
- **Non-touching Loops:** Loops that have no vertex in common (and, therefore, no edge.)

Example

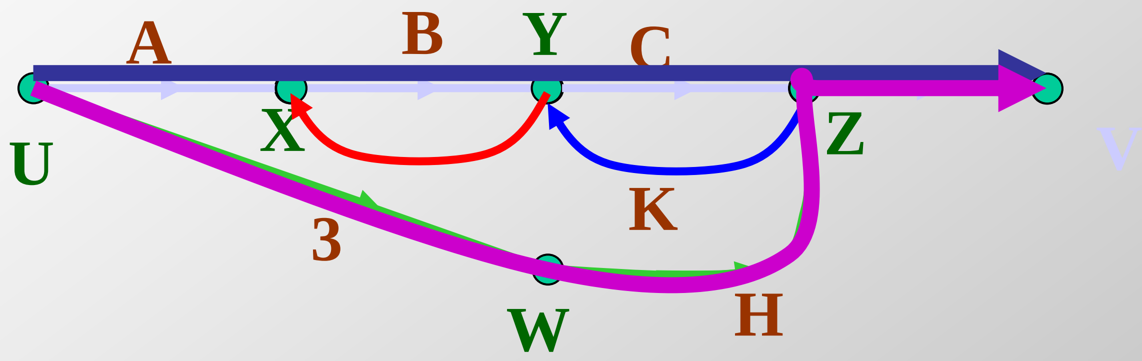
Determine the transfer function between V and U



- The number of forward paths from U to V = ?
- Path Gains ?
- Loops ?
- Determinant ?
- Cofactors ?
- Transfer function ?

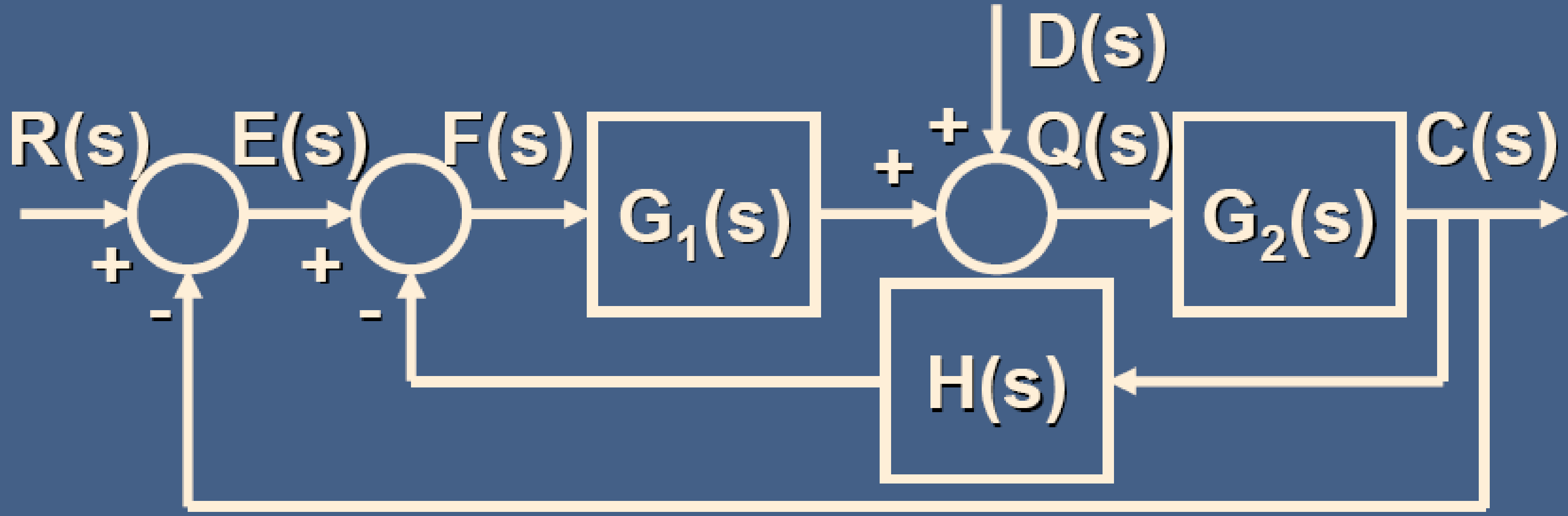
Example

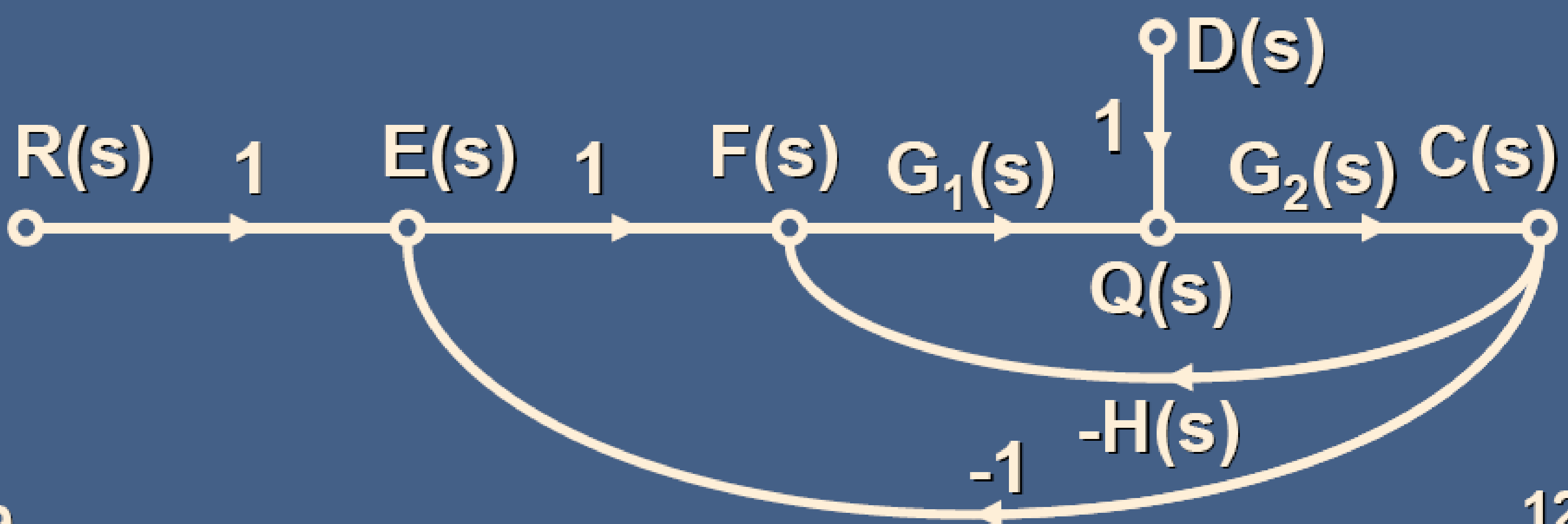
Determine the transfer function between V and U



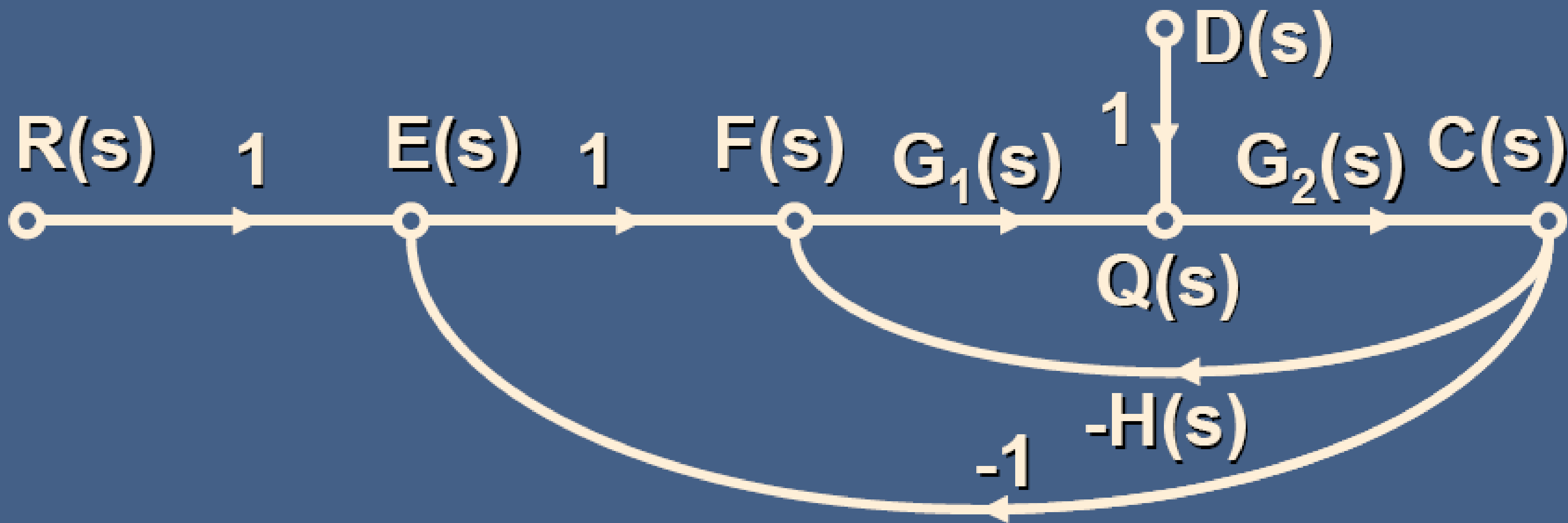
- The number of forward paths from U to V = 2
- Path Gains ABC, 3H
- Loop Gains B, CK
- Transfer function (ABC+3H-3HB)/(1-B-CK)

Signal Flow Graph Example



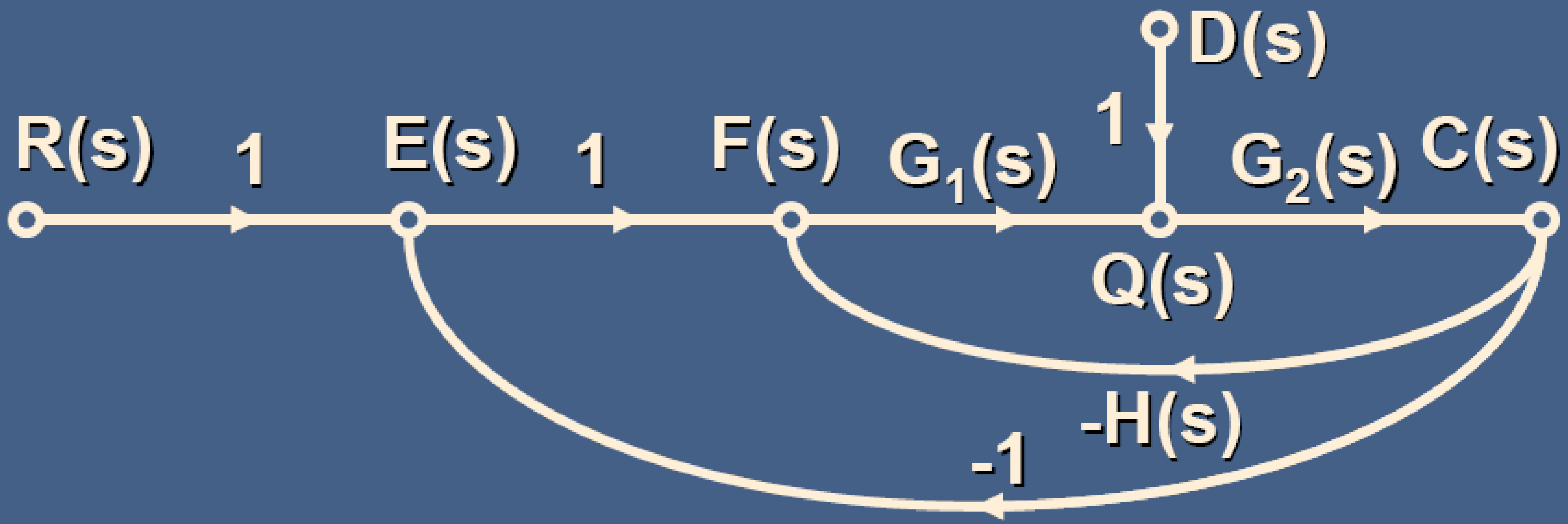


Example of Gain Formula Use



- Assume R(s) = 0, desire to find the transfer function C(s)/D(s).
- There is only one forward path between D(s) and C(s), therefore k = 1.
- There are two loops. They are touching.

Example of Gain Formula Use

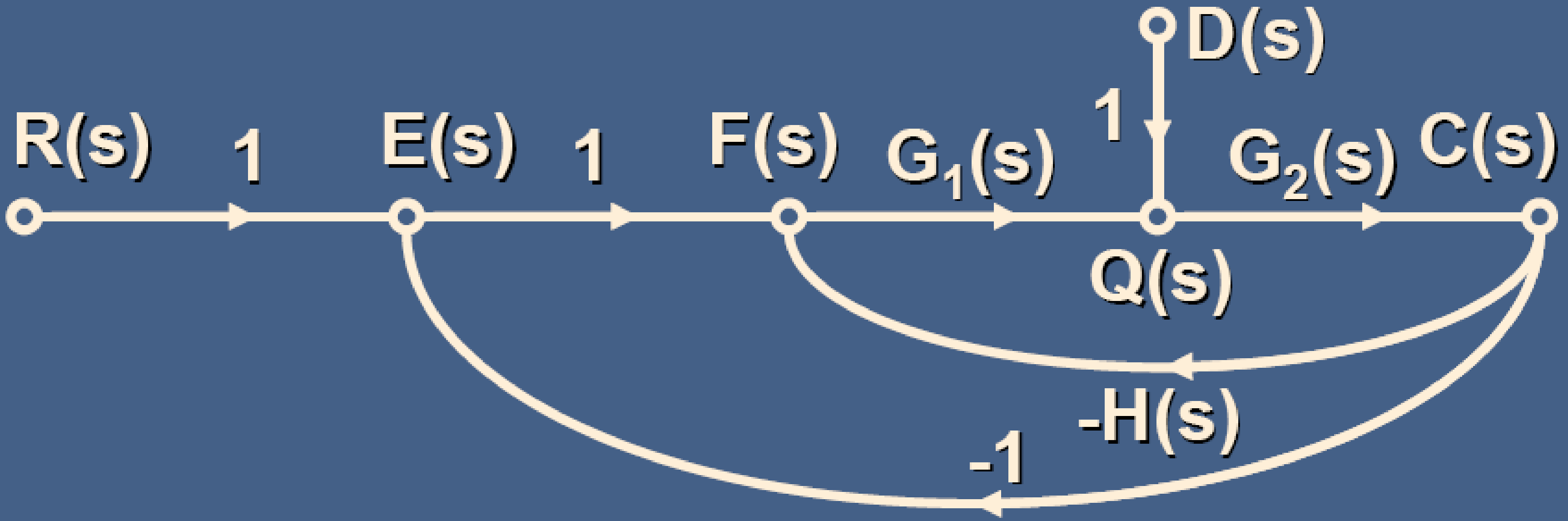


$P_1 = G_2(s)$

$\Delta = 1 - [-G_1(s)G_2(s)H(s) - G_1(s)G_2(s)]$

$\Delta_1 = 1$ (Both loops touch the k^{th} path)

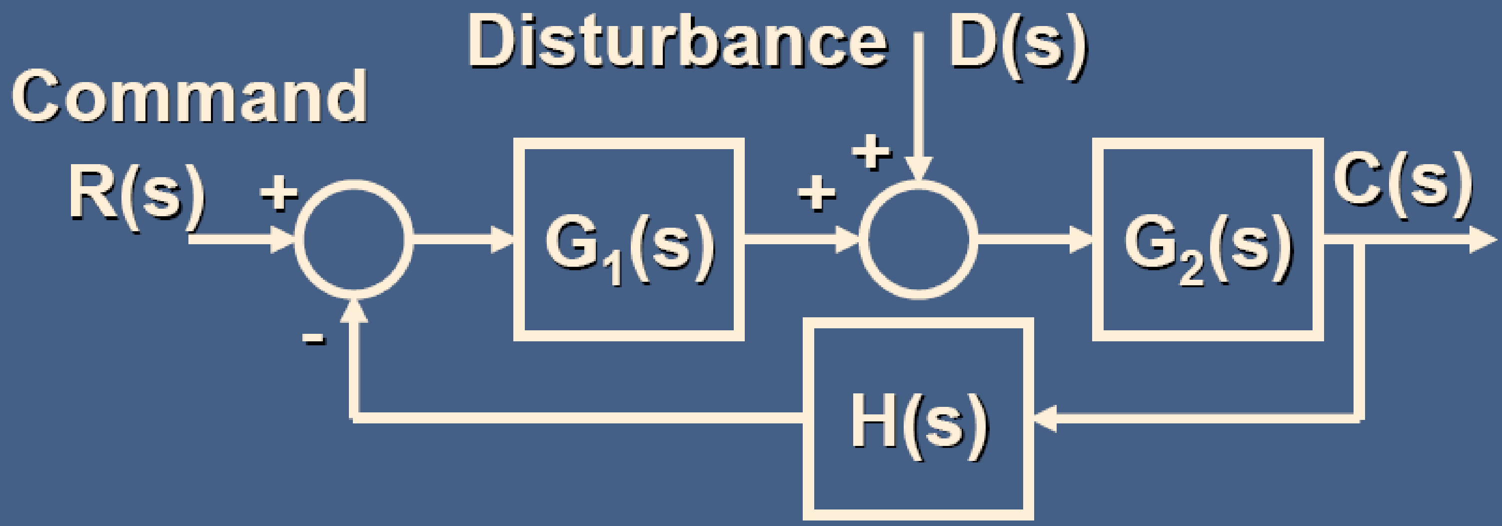
Example of Gain Formula Use



$$\frac{C(s)}{D(s)} = \sum_k \frac{P_k \Delta_k}{\Delta}$$

$$\frac{C(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s) + G_1(s)G_2(s)}$$

Block Diagram with Disturbance Input

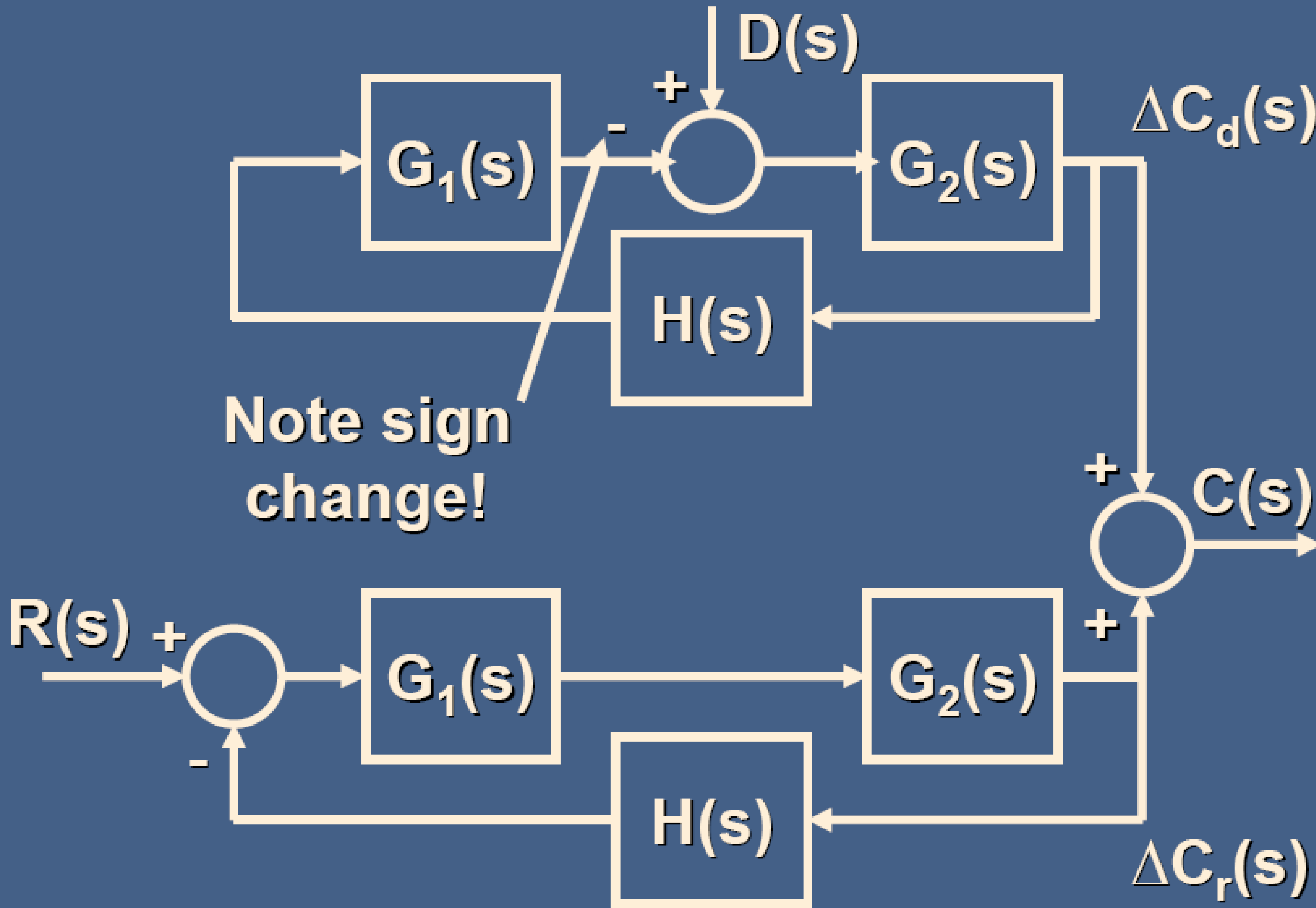


- A disturbance input is an unwanted or unavoidable input signal that affects a system's output. Examples:

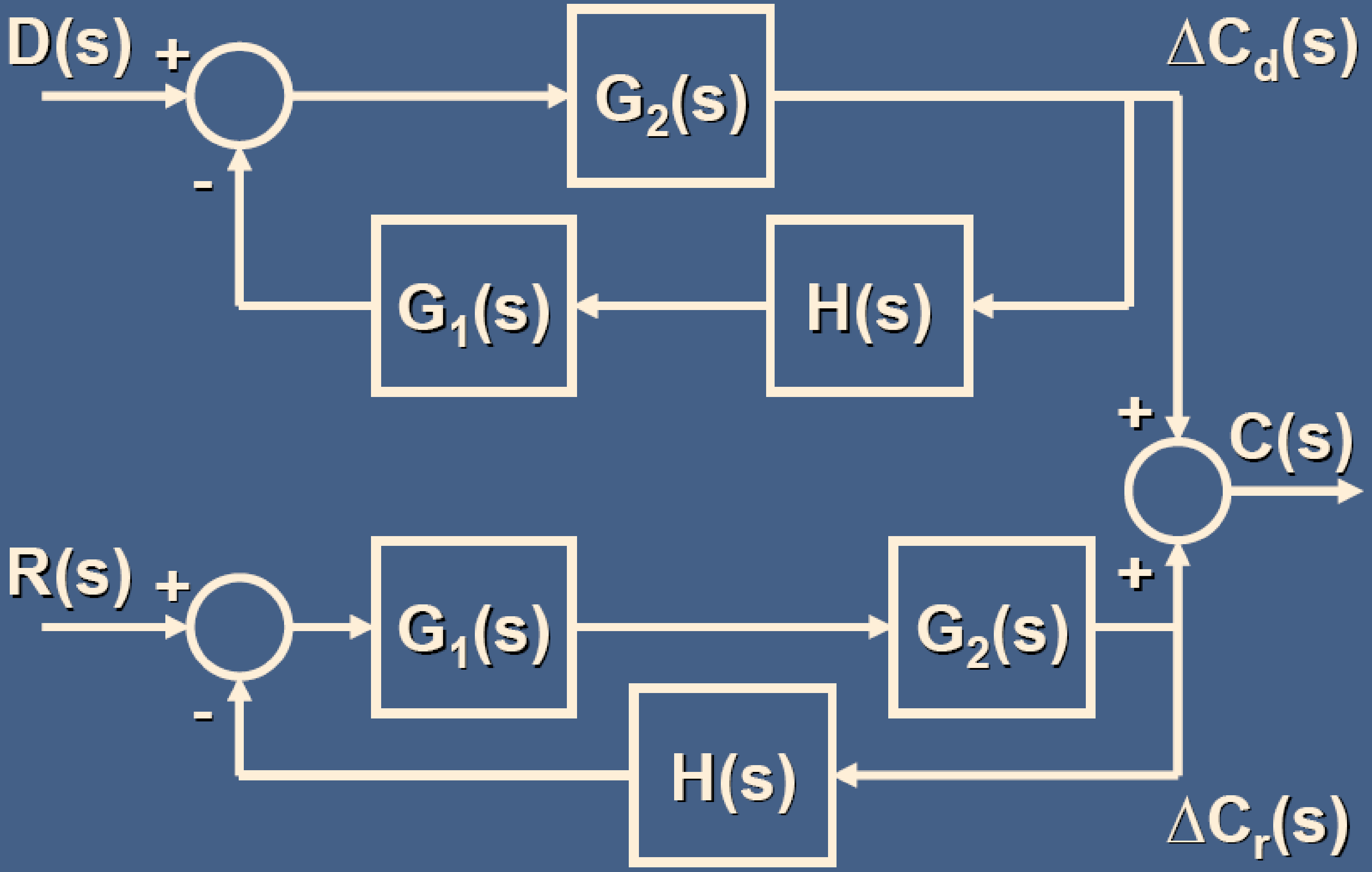
  - load torque in motor control

  - open door in room climate control

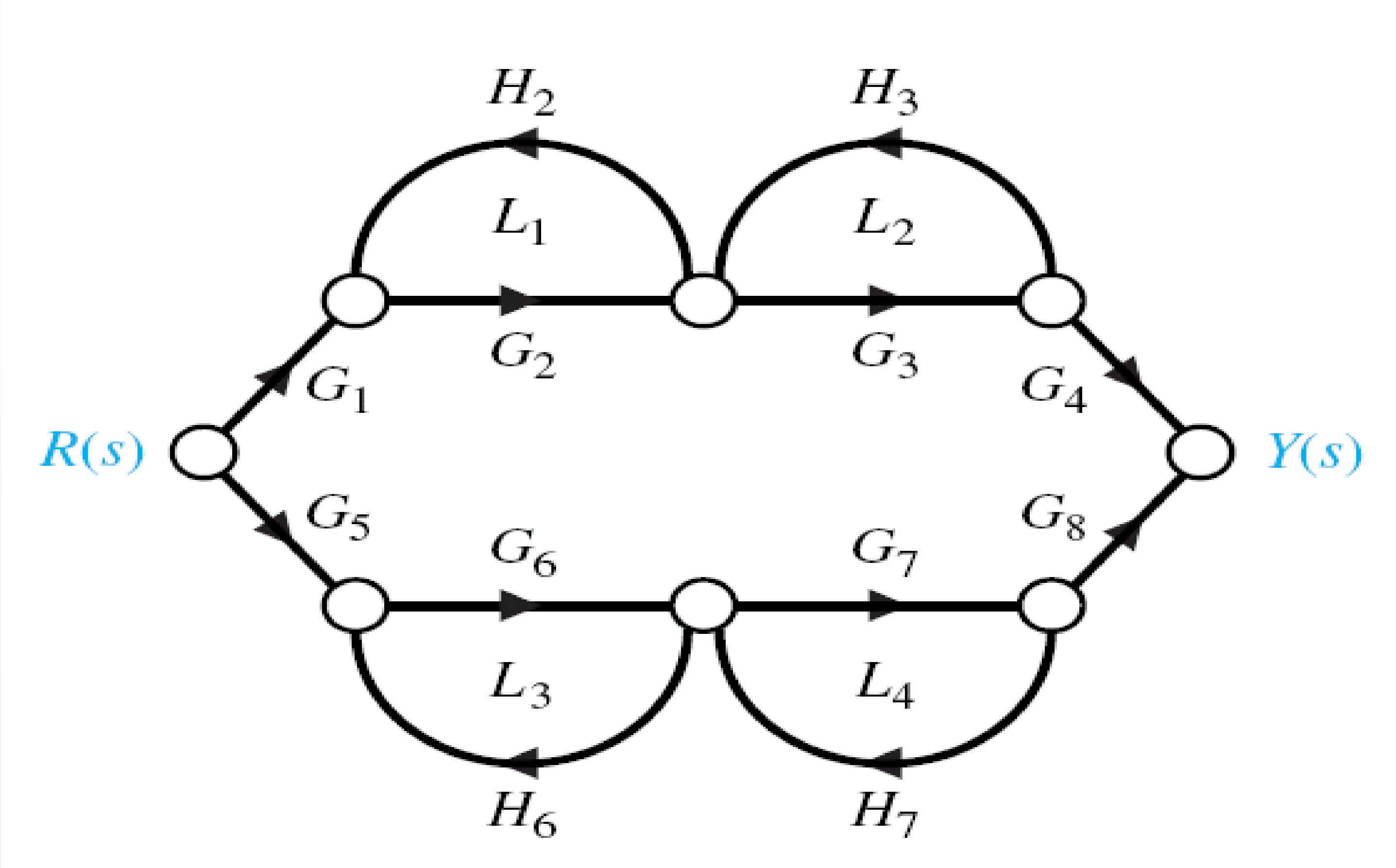
Superposition: $C(s) = \Delta C_r(s) + \Delta C_d(s)$



Disturbance Portion Redrawn



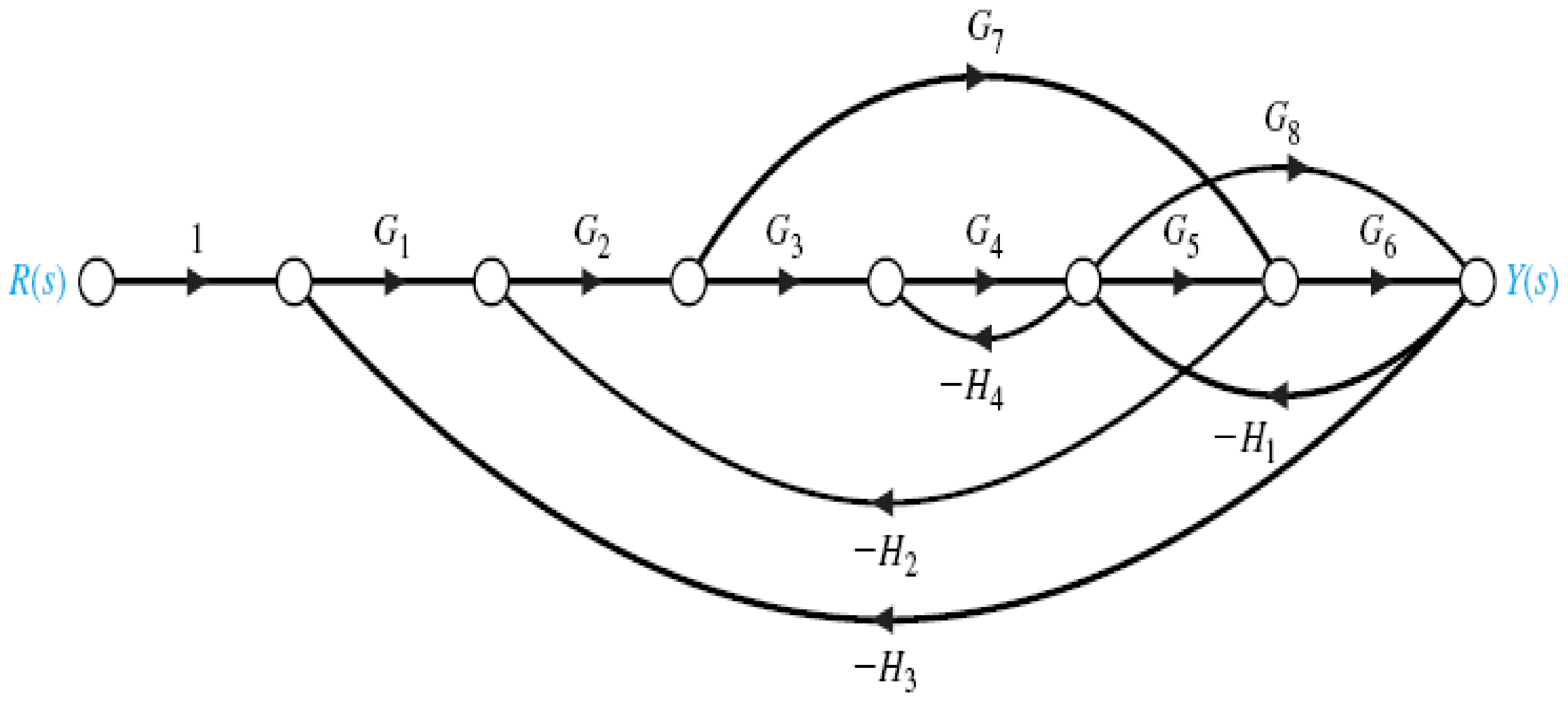
An example



An example:

- Two paths :P1, P2
- Four loops
- $P_1 = G_1G_2G_3G_4$, $P_2 = G_5G_6G_7G_8$
- $L_1=G_2H_2$ $L_2=G_3H_3$ $L_3=G_6H_6$ $L_4=G_7H_7$
- $\Delta = 1 - (L_1+L_2+L_3+L_4)+(L_1L_3+L_1L_4+L_2L_3+L_2L_4)$
- Cofactor for path 1: $\Delta 1 = 1 - (L_3+L_4)$
- Cofactor for path 2: $\Delta 2 = 1-(L_1+L_2)$
- $T(s) = (P_1\Delta_1 + P_2\Delta_2)/\Delta$

Another example



- 3 Paths
- 8 loops

Mason's Gain Rule (1956)

Given an SFG, a source and a sink, N forward paths between them and K loops, the gain (transfer function) between the source-sink pair is

$$T_{ij} = \frac{\sum P_k \Delta_k}{\Delta}$$

P_k is the gain of path k, Δ is the "graph determinant":

$\Delta = 1 - \sum$(all loop gains)

$+ \sum$(products of non-touching-loop gain pairs)

$- \sum$(products of non-touching-loop gain triplets)

$+ \ldots$

$\Delta_k = \Delta$ of the SFG after removal of the k_{th} forward path

Terms for Mason's Gain Formula

- Path: A branch or sequence of branches that can be traversed from one node to another.
- Loop: A closed path, along which no node is met twice, that originates and terminates in the same node.
- Nontouching: Two loops are nontouching if they do not share a common node.
- Gain: Refers, in this case, to the product of transfer functions.

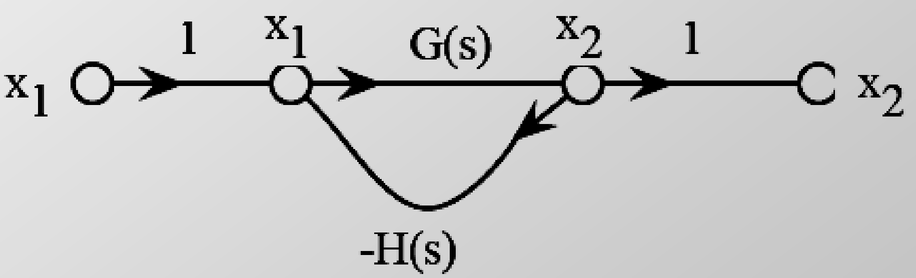
Mason's Gain Formula: $\frac{O(s)}{I(s)} = \sum_k \frac{P_k \Delta_k}{\Delta}$

- P_k = the gain of the k^{th} forward path between I(s) and O(s).
- Δ = 1 - (sum of all individual loop gains)
  + (sum of gain products of all combinations of 2 nontouching loops)
  - (sum of gain products of all combinations of 3 nontouching loops)
  + …
- Δ_k = value of Δ for that part of graph nontouching the k^{th} forward path.

Mason's Rule for Simple Feedback loop

$P_1 = G(s)$

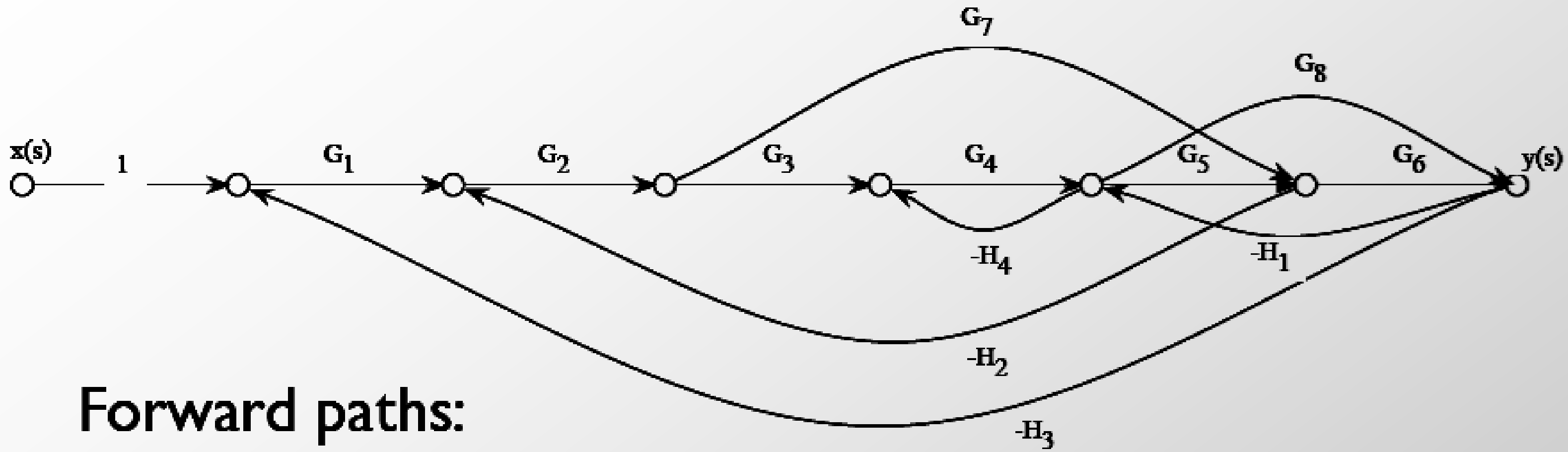
$L_1 = -G(s)H(s)$



$\Delta = 1 - (-G(s)H(s))$

$\Delta_1 = 1$

$$T(s) = \frac{P_1 \Delta_1}{\Delta} = \frac{G(s)}{\Delta} = \frac{G(s)}{1+G(s)H(s)}$$



Forward paths:

$P_1 = G_1G_2G_3G_4G_5G_6$ $P_2 = G_1G_2G_7G_6$ $P_3 = G_1G_2G_3G_4G_8$

Feedback loops:

$L_1 = -G_2G_3G_4G_5H_2$ $L_2 = -G_5G_6H_1$ $L_3 = -G_8H_1$

$L_4 = -G_7H_2G_2$ $L_5 = -G_4H_4$ $L_6 = -G_1G_2G_3G_4G_5G_6H_3$

$L_7 = -G_1G_2G_7G_6H_3$ $L_8 = -G_1G_2G_3G_4G_8H_3$

Loops $\{3,4\},\{4,5\}$ and $\{5,7\}$ don't touch

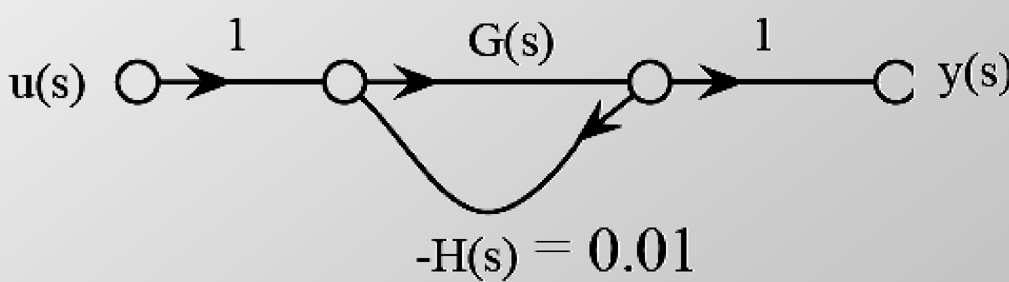
$\Delta = 1-(L_1+L_2+L_3+L_4+L_5+L_6+L_7+L_8)+(L_3L_4+L_4L_5+L_5L_7)$

$\Delta_1 = \Delta_3 = 1$, $\Delta_2 = 1-L_5 = 1 - G_4H_4$

$$T(s) = \frac{y(s)}{x(s)} = \frac{P_1+P_2\Delta_2+P_3}{\Delta}$$

A Feedback Loop Reduces Sensitivity To Plant Variations

y(s)/u(s)=G/(1+GH)



G=10000

y(s)/u(s)=10000/(1+10000*0.01)=99.01

G=20000

y(s)/u(s)=20000/(1+20000*0.01)=99.50